

## Collateral restrictions and liquidity under-supply: a simple model

Ana Fostel · John Geanakoplos

Received: 14 August 2006 / Revised: 18 April 2007 / Published online: 7 June 2007  
© Springer-Verlag 2007

**Abstract** We show that very little is needed to create liquidity under-supply in equilibrium. Credit constraints on *demand* by themselves can cause an under-supply of liquidity, without the uncertainty, intermediation, asymmetric information or complicated international financial framework used in other models in the literature. We show that the under-supply is a non-monotone function of the demand distortion that causes it, a result that may have interesting implications for emerging markets economies. Finally, when we make the credit constraint endogenous, the inefficiency can be large due to the presence of a multiplier.

**Keywords** Liquidity under-supply · Credit constraint · Non-monotonicity · Multiplier · Collateral equilibrium

**JEL Classification Numbers** D51 · E44 · F30 · G15

---

The authors thank the very helpful comments of Andres Velasco, Herbert Scarf and all the participants of the Mathematical Economics Seminar at Yale.

---

A. Fostel (✉)  
Department of Economics, George Washington University,  
710 21st St., NW, Washington, DC 20052, USA  
e-mail: afostel@gwu.edu

J. Geanakoplos  
Department of Economics, Yale University, Box 208281,  
06520-8281 New Haven, CT, USA  
e-mail: john.geanakoplos@yale.edu

J. Geanakoplos  
Santa Fe Institute, 1399 Hyde Park Road, Santa Fe,  
New Mexico 87501, USA

## 1 Introduction

Liquidity has been defined in many different ways. We will adopt two definitions of liquidity, which we call Physical Liquidity and Financial Liquidity. They refer to the flexibility to move physical goods (money) across different projects (investments). The goal of this paper is to explain liquidity under-supply in equilibrium: when firms, in a decentralized way, optimally choose their own liquidity positions, the economy as a whole ends up with less liquidity than the second best efficient level.

There are already many explanations in the literature for the liquidity under supply phenomenon. For [Holmstrom and Tirole \(1998\)](#), liquidity is related to how complete the asset markets are, and in particular, with the ability of the private sector to buy a riskless asset in order to transfer wealth across time. With this definition, they get under-supply of liquidity in the presence of aggregate uncertainty. (If the only tradable security is assumed to be shares in aggregate production, the riskless asset will be missing.) For [Kiyotaki and Moore \(2000\)](#), liquidity is money, and they obtain liquidity shortages in equilibrium in an infinite horizon economy under non-saleability assumptions. For [Morris and Shin \(2003\)](#), liquidity is the thickness of markets and they use global games techniques to show how liquidity under-provision can arise from asymmetric information. There is a big literature, including [Diamond and Rajan \(2001\)](#) and [Chang and Velasco \(1999\)](#), in which liquidity shortages and crises arise from bank intermediation and failures. For [Geanakoplos \(2003\)](#), liquidity declines when lenders endogenously raise margin requirements. Finally, [Caballero and Krishnamurthy \(2001\)](#) argue that liquidity under provision is a characteristic of emerging market economies. They work in a model with two different liquidities (a domestic and an international), idiosyncratic and aggregate uncertainty, and credit constraints in both domestic and international markets.

In this paper, we use a simple model to show that a *sufficient* condition to get liquidity under-supply is the presence of credit constraints on *demand*. In particular, we do not need any kind of uncertainty, asymmetric information, intermediation or a complicated international financial framework to prove the result. Our model also shows that the aggregate under-supply of liquidity is not necessarily a characteristic of emerging market economies, since the inefficiency can arise in mature markets as well, as long as there are credit constraints.

The liquidity under-supply we describe is a particular example of a more general principle that we call the Offsetting Distortions principle. This principle states that a distortion in *demand* for any good can often be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. In our model, credit constraints play the role of demand distortions, which, as the Offsetting Distortion principle suggests, can be understood as an inefficiency in the supply of liquidity.

Next, we explore the relationship between the liquidity under-supply and the demand distortion that creates it. We find that the liquidity under-supply is a *non-monotone* function of the credit constraint. If we interpret the credit constraint as the degree of financial development in the economy, our second proposition states that the liquidity under-supply is a non-monotone function of the degree of financial development. This suggests that when financial markets are very undeveloped, financial

innovation may paradoxically make government intervention (like taxation of illiquid investments) more necessary. It is difficult not to think about the financial innovation and simultaneous, dramatic, reduction of government participation in financial markets that took place in Latin American economies during the 1990s. Numerous liquidity crises occurred in these economies during that period. The non-monotonicity occurs for any commodity if we suppose that the central planner can affect only part of the supply curve: Second Best inefficiency is then a non-monotone function of the demand distortion. The nature of liquidity as flexibility insures that the central planner can indeed only affect part of the supply curve, thus linking liquidity with non-monotone distortions.

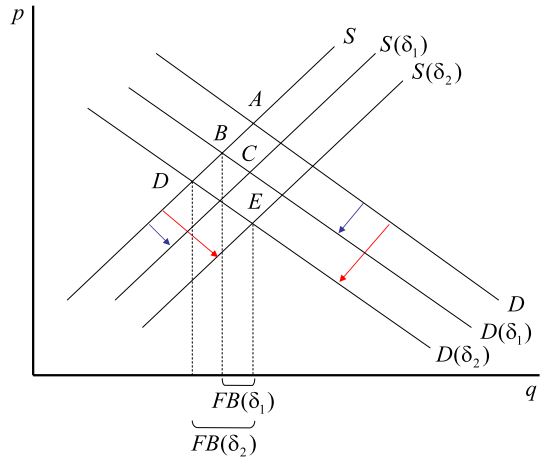
In the last part of the paper we think about the *magnitude* of the liquidity under-supply. We model the credit constraint by assuming that borrowers will default unless their promises are covered by collateral. Further, we assume that only an exogenous proportion  $\beta$  of one durable good can serve as collateral. This parameter  $\beta$  will represent the degree of financial development of the economy. We show that the magnitude of the under-supply can be larger when the price of the collateral is endogenous, giving rise to a Liquidity Under-Supply Multiplier. Any policy intervention that affects the interest rate in equilibrium will have two effects on the borrowing constraint: a direct effect, also present in the case when the credit constraint is exogenous, and an indirect effect through the price of the collateral.

Next, we explore our findings in a particular example in which utilities for the consumption good and the collateral are quadratic. In this context, we can be more precise about the effects on liquidity of the degree of financial development,  $\beta$ , and the marginal utility of collateral,  $\lambda$ . First, in economies where the market value of collateral is low (because  $\lambda$  is low), there will always be liquidity under-supply, no matter how high  $\beta$  is. The government should intervene, for example, by taxing illiquid investments. Paradoxically, as  $\beta$  increases and the financial system becomes more developed, the need for government intervention (taxation) increases. For higher values of  $\lambda$ , the liquidity under-supply is non-monotonic in  $\beta$ , increasing for low values and decreasing for high values of  $\beta$ . Similarly, the liquidity under-supply multiplier is increasing in  $\beta$  for low values of  $\lambda$  and becomes non-monotonic in  $\beta$  for high values of  $\lambda$ .

Finally, in the last two sections we show that the results continue to hold in a stationary equilibrium in an Overlapping Generations Model or in a model with uncertainty.

The paper is organized as follows. In Sect. 2 we present diagrams explaining the general principles of Offsetting Distortions and Non-Monotone Second Best Inefficiency. In Sect. 3 we briefly discuss different definitions of liquidity used in the literature as well as the definitions we will consider. Section 4 presents the model and shows the liquidity under-supply result. In Sect. 5 we show that the liquidity under-supply is non-monotone in the credit constraint. In Sect. 6 we endogenize the credit constraint and prove the existence of the liquidity under supply multiplier. Also we solve and simulate the quadratic example. In Sect. 7 we note that all the preceding results continue to hold in a stationary equilibrium in an Overlapping Generations Model. Section 8 shows that adding uncertainty does not cause any qualitative changes, but it does increase the magnitude of the under-supply. All proofs are presented in the Appendix.

**Fig. 1** Offsetting distortions and first best monotonicity



## 2 Offsetting distortions and non-monotone inefficiencies

### 2.1 Offsetting distortions principle

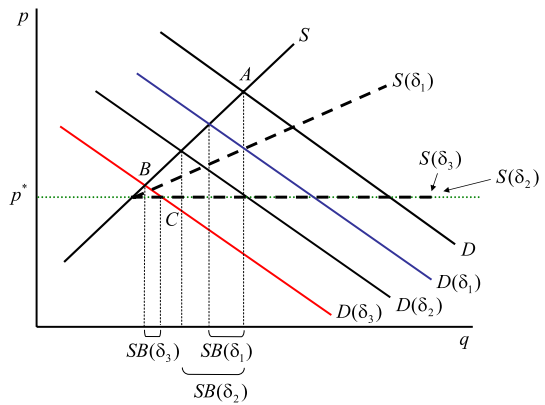
Consider a market with its demand,  $D$ , supply,  $S$ , and initial equilibrium at  $A$  as shown in Fig. 1. There are no distortions of any kind in this market and hence the equilibrium is First Best efficient. Suppose now that we introduce some distortion to demand, say  $\delta_1$ . The new demand is  $D(\delta_1)$  instead of  $D$  and the new equilibrium price and quantities are smaller at point  $B$ . Given the demand distortion, the reduction in quantity can be interpreted as a supply inefficiency since a central planner could compensate for the demand dislocation by introducing a distortion to supply, shifting the curve to  $S(\delta_1)$ . At the new equilibrium  $C$ , the quantity is restored to its original First Best level, though the equilibrium price is lower than before.

The Offsetting Distortions principle<sup>1</sup> states that a distortion in *demand* can often be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. If the central planner knows he cannot restore demand, then he must regard supply as in need of stimulation.

In this paper we use the Offsetting Distortions principle to see how an aggregate under-supply of liquidity can arise from inefficiencies in the demand for credit. A collateral restriction that reduces the effective demand for loans can be interpreted as an inefficiency in the supply of loans. In effect, the collateral restriction on demand for loans manifests itself as an under-supply of liquidity, since loans can only be offered out of a pool of liquid capital created in the previous period.

<sup>1</sup> The Offsetting Distortions principle is not novel. The idea, that a distortion on demand calls for a perturbation of supply, reminds us of what has been known as the Second Best Theory developed by Lancaster and Lipsey during the 1950s.

**Fig. 2** Offsetting distortions and second best non-monotonicity



### 2.2 Non-monotone inefficiencies

We take for granted that the demand distortion cannot be undone, and focus attention on the inefficiency of supply. Define the First Best Inefficiency as the difference between the First Best quantity (which could be restored if supply perturbations were unconstrained), and the quantity supplied in equilibrium under demand distortions but before intervention. For instance, in Fig. 1 the First Best Inefficiency associated with the demand distortion  $\delta_1$ , what we denoted by  $FB(\delta_1)$ , is the difference between the quantities at  $C$  (or  $A$ ) and  $B$ . Consider now a larger demand distortion, say  $\delta_2 > \delta_1$ . The First Best inefficiency associated to  $\delta_2$ ,  $FB(\delta_2)$ , is given by the difference in quantities at  $E$  (or  $A$ ) and  $D$ . Clearly,  $FB(\delta_1) < FB(\delta_2)$ . The First Best Inefficiency is a monotone function of the demand distortion.

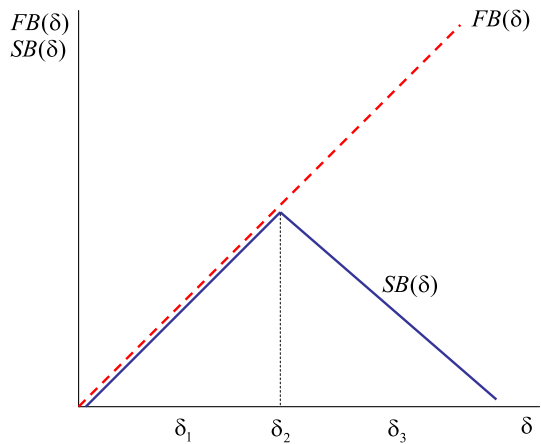
Suppose, however, that the central planner is constrained in how he can perturb supply. For example, suppose he cannot increase supply for prices below a lower limit of  $p^*$ . The equilibrium quantity attained after the planner intervenes optimally, but subject to this constraint, will be called the Second Best quantity.

Define the Second Best Inefficiency as the difference between the Second Best quantity and the equilibrium quantity attained without intervention. Surprisingly, the Second Best Inefficiency is not monotone in the demand distortion. As we can see in Fig. 2, if the demand distortion were small enough, such as  $\delta = \delta_1$ , in fact for all  $0 < \delta < \delta_2$ , the central planner would be able to restore the quantity all the way back to its first best level. Thus, the Second Best inefficiency is equal to the First Best Inefficiency, and grows with  $\delta$ .

But for  $\delta > \delta_2$ , the Second Best Inefficiency declines as  $\delta$  increases. For instance, consider the demand distortion  $\delta_3$ , as in Fig. 2. Given the constraint faced by the central planner, it is clear that the best he can do is to perturb supply to  $S(\delta_3)$  resulting in a Second Best quantity at point  $C$ . Although this quantity is bigger than the one at  $B$ , it is far below the first best level at  $A$ , and not much bigger than at  $B$ .

A last observation is that the Second Best Inefficiency becomes bigger when the demand curve becomes flatter. We will turn to this point at the end of the paper.

**Fig. 3** First best and second best inefficiency



In Fig. 3 we can see the different behavior of the First Best and Second Best Inefficiencies. The dotted curve represents the First Best Inefficiency as a function of the distortion  $\delta$ , while the full one represents the Second Best Inefficiency. While the First Best Inefficiency is a monotone function of the demand distortion, the Second Best Inefficiency is not.

### 3 Defining liquidity

Liquidity, and the closely related notion of flexibility, are intuitively understood by economists and others. However, when one tries to distil these notions into precise definitions, one finds that liquidity has been defined in many different ways.

According to some authors, like [Shubik \(1999\)](#) or [Kiyotaki and Moore \(2000\)](#), liquidity refers to a substance, like gold, which is accepted as a means of payment. An illiquid agent who is very rich in other goods may not be able to make purchases, at least for the moment, because he lacks gold or cash.

A second definition of liquidity refers to the thinness of the market for some good. An agent is in possession of an illiquid good if he cannot quickly sell it without a big discount. Clearly, this definition tries to capture the idea of flexibility mentioned before. It is about speed of reaction and about how costly it is to change financial position if needed. This is the most common notion used by market participants. It is also often used as a definition in academics as in [Diamond \(1986\)](#), [Jones and Ostroy \(1984\)](#), and [Morris and Shin \(2003\)](#).

A third definition of liquidity stresses the completeness of markets. [Holmstrom and Tirole \(1998\)](#) define liquidity as the availability of instruments that can be used to transfer wealth across periods. An economy is more liquid than another if it has more markets. These authors particularly emphasize the ability of private agents to purchase a variety of assets that transfer wealth to the future. They suggest that the government can increase liquidity by creating and selling debt, which the private agents can then hold as a hedge against production emergencies.

A fourth definition is what we call financial liquidity. How liquid an agent is depends on his ability to borrow against the present value of his future income, that is, to sell contingent promises of future deliveries. Financial liquidity depends not only on the presence of various contingent promises, but also on the ability of agents to credibly commit to honor these promises, say by issuing collateral. This notion of liquidity is the one used in Geanakoplos (1997, 2003), Diamond and Rajan (2001), and Caballero and Krishnamurthy (2001).

Finally, the last notion is what we call physical liquidity. As the name suggests, this notion refers to the flexibility to move goods across different projects. A project is liquid if an investor can move his physical inputs to another project as easily as he could if he had kept them in storage.

All these definitions try to capture in a different way the idea of flexibility: flexibility to make transactions, flexibility to exit a market and change portfolio composition quickly and without cost, flexibility to move wealth across time given that agents can buy or sell contingent promises and finally, flexibility to move goods across different activities or projects.

In this paper we will focus on the last two notions of liquidity: financial liquidity and physical liquidity. We shall find that when agents take liquidity decisions in a decentralized way, the economy as a whole will produce too little liquidity. In the next sections we will use the general principles of Offsetting Distortions and Non-Monotonicity of the Second Best Inefficiency to understand financial and physical liquidity under-supply as well as the non-monotone behavior of these inefficiencies.

## 4 Liquidity under-supply

### 4.1 The model

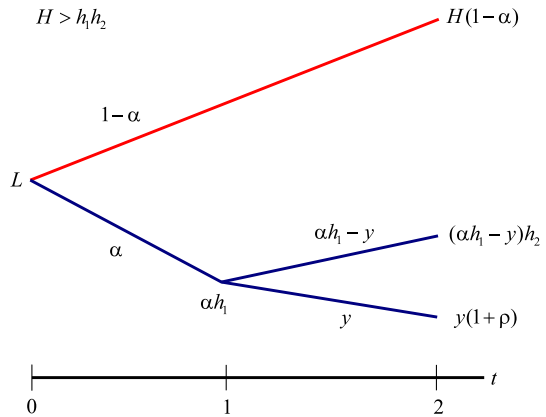
Consider an economy in which there are three periods,  $t = 0, 1, 2$ , and a single consumption good which is durable, and hence serves as a store of value as well as for productive investment. All consumption takes place in the last period. There is a continuum of firms of two types: firms of type  $L$ , lenders, and firms of type  $B$ , borrowers.

At  $t = 0$  a type  $L$  firm is endowed with a single unit of the good. He has two investment options. The first is a long-term (illiquid) investment that has a constant gross return of  $H$  per unit of investment at  $t = 2$ , while the second is short-term (liquid) and pays  $h_1 < H$  at  $t = 1$  per unit of investment. At stage 0 firm  $L$  decides on the percentage  $\alpha \in [0, 1]$  of the good to invest in the short-term project.

An  $L$  firm arrives at period 1 with  $\alpha h_1$  wealth from his short-term investment. At this point he has to decide how much to lend,  $y$ , to a type  $B$  firm, from which he gets a payoff of  $y(1 + \rho)$  at  $t = 2$ , where  $\rho$  is the market interest rate. There exists another investment option which pays  $h_2$  per unit of investment at  $t = 2$ . The decisions faced by an  $L$  firm at different periods can be seen in Fig. 4.

Type  $B$  firms only play a role at  $t = 1$ . Each  $B$  firm has an investment opportunity with a gross return per unit invested of  $R > 1$  at  $t = 2$ . We will assume that the return of this investment is extremely good, i.e.,  $Rh_1 > H > h_1h_2$ . Each  $B$  firm has no

**Fig. 4** Firm  $L$ 's decisions



endowment and therefore chooses to borrow  $x$  from type  $L$  firms. However, each  $B$  firm is credit constrained, i.e., the amount she can borrow has to satisfy,

$$(1 + \rho)x \leq \Pi$$

where  $\Pi$  is an exogenous limit to what the firm can promise.<sup>2</sup> This constitutes the only market imperfection present in the economy.

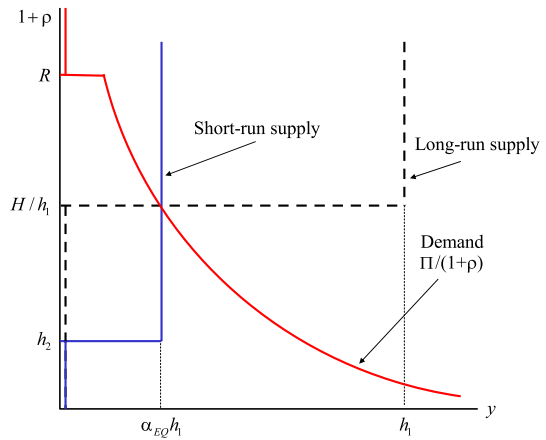
Finally, at period  $t = 2$  debts are paid back and consumption is realized. Agents in this economy care only about output in  $t = 2$ . There is no uncertainty of any kind.

In this model, the measure of liquidity is given by  $\alpha$ . The short-term investment gives a firm of type  $L$  the flexibility at  $t = 1$  to reinvest the physical good into two different new projects: he can invest in a new project that returns  $h_2$  at  $t = 2$  or he can decide to enter in the credit market which would give him a return of  $1 + \rho$ . On the other hand, the portion of the initial investment devoted to the long-term investment,  $1 - \alpha$ , gives him a return of  $H$  at  $t = 2$  but no flexibility at all at  $t = 1$  to move those physical goods to other projects. With this interpretation  $\alpha$  is a measure of physical liquidity. Suppose now that  $L$  firms at time 0 are endowed with a certain amount of cash instead. They can buy a long-term asset that pays  $H$  at  $t = 2$ . However, once in this position, they cannot sell any promise at  $t = 1$  using as collateral the present value of their future income  $H$ . On the other hand, they can invest a proportion  $\alpha$  of their initial cash holdings on a short-term investment with a return of  $h_1$  at  $t = 1$ . In the second period they have two financial options, invest with a return of  $h_2$  or enter into the credit market which will yield a return of  $1 + \rho$ . Clearly, with this interpretation  $\alpha$  now becomes a measure of financial liquidity.

<sup>2</sup> It will be crucial that the amount  $x$  that can be borrowed is a decreasing function of the interest rate  $\rho$ . This is perfectly natural since the promised payments are not delivered until the next period. This kind of credit constraint is standard in the literature. There are many different micro foundations stories that justify it; moral hazard problems, the need of collateral, etc. In the last part of the paper we will interpret the credit constraint in terms of collateral restrictions.



**Fig. 5** Equilibrium in the case  $\Pi < h_1 h_2$



At  $t = 0$ ,  $L$  firms face the option of a liquid investment or an illiquid one. The illiquid investment has a higher return than the liquid investment, but the latter allows firms  $L$  to become lenders at  $t = 1$ .

Since every agent cares only about output in period 2, any Pareto efficient allocation would maximize period 2 output. Hence, we take total output in period 2 as our measure of welfare. Since  $Rh_1 > H$ , the first best thing to do from a social point of view is to invest everything in the liquid option,  $\alpha = 1$ , and then lend it all,  $y = h_1$ , to the  $B$  firms. As we will see, this will not be the outcome in equilibrium.

Let us be more precise about all this.

**Definition 1** An *equilibrium* in this economy consists of decisions  $(\alpha_{EQ}, y_{EQ}, x_{EQ})$  and a price,  $\rho_{EQ}$  such that,

(a)  $L$  firms choose  $\alpha_{EQ}$  in period 0 and  $y_{EQ}$  in period 1 such that  $(\alpha_{EQ}, y_{EQ})$  solves

$$\begin{cases} \max_{\alpha, y} (1 - \alpha)H + (\alpha h_1 - y)h_2 + y(1 + \rho_{EQ}) \\ \text{s.t. } 0 \leq \alpha \leq 1 \\ \quad 0 \leq y \leq \alpha h_1 \end{cases}$$

(b)  $B$  firms choose  $x_{EQ}$  in period 1 such that  $(x_{EQ})$  solves

$$\begin{cases} \max_x Rx - (1 + \rho_{EQ})x \\ \text{s.t. } 0 \leq (1 + \rho_{EQ})x \leq \Pi \end{cases}$$

(c)  $x_{EQ} = y_{EQ}$ .

Firms maximize consumption at period 2 taking the price as given and markets clear. We have assumed  $0 < h_1 h_2 < H < Rh_1$ . The nature of the equilibrium depends on whether  $\Pi < H$ ,  $H < \Pi < Rh_1$  or  $\Pi \geq Rh_1$ . The equilibrium in this model can be seen in Fig. 5 (in the case  $\Pi < h_1 h_2$ ).

The decreasing curve

$$D(\rho) = \begin{cases} 0, & 1 + \rho > R \\ \left[0, \frac{\Pi}{1+\rho}\right], & 1 + \rho = R \\ \frac{\Pi}{1+\rho}, & 1 + \rho < R \end{cases} \tag{1}$$

represents the constrained demand of the *B* firms for loans *x*.

The *L* firms will put all the good at  $t = 0$  into the illiquid investment if  $H > h_1(1 + \rho)$ , and all the good into the liquid investment if  $H < h_1(1 + \rho)$ . Once they have invested  $\alpha$  into the liquid investment, they will lend all  $\alpha h_1$  if  $1 + \rho > h_2$ . The long-run supply of *y* is thus

$$LS(\rho) = \begin{cases} 0, & 1 + \rho < H/h_1 \\ [0, h_1], & 1 + \rho = H/h_1 \\ h_1, & 1 + \rho > H/h_1 \end{cases} \tag{2}$$

The short-run supply of *y* is

$$SS(\rho, \alpha) = \begin{cases} 0, & 1 + \rho < h_2 \\ [0, \alpha h_1], & 1 + \rho = h_2 \\ \alpha h_1, & 1 + \rho > h_2 \end{cases} \tag{3}$$

The dotted and filled curves represent, respectively, the long- and the short-run supply of loans *y* by the *L* firms.

When  $0 < \Pi < H$  the decreasing part of the demand curve cuts the supply curve on its horizontal segment, as shown in Fig. 5. In that case the equilibrium is  $1 + \rho_{EQ} = H/h_1$ ,  $y_{EQ} = x_{EQ} = \Pi h_1/H$  and  $\alpha_{EQ} = \Pi/H$ . Clearly, equilibrium is not First Best efficient, since *B* firms borrow  $\Pi h_1/H < h_1$ , and output is  $H(1 - \Pi/H) + R h_1 \Pi/H < R h_1$ .

When  $\Pi \geq R h_1$  the demand curve cuts the vertical part of the supply curve where  $y = h_1$  and the equilibrium is  $1 + \rho_{EQ} = R$ ,  $y_{EQ} = x_{EQ} = h_1$  and  $\alpha_{EQ} = 1$ . For  $H \leq \Pi \leq R h_1$ , the downward sloping piece of the demand curve cuts the vertical part of the supply curve where  $y = h_1$ . The equilibrium quantity is still First Best, although the interest rate is smaller:  $H/h_1 < 1 + \rho_{EQ} < R$ ,  $y_{EQ} = x_{EQ} = h_1$  and  $\alpha_{EQ} = 1$ .

Figure 5 illustrates the case where  $\Pi < h_1 h_2 < H$ , so that demand cuts not only the horizontal line  $(1 + \rho) = H/h_1$ , but also the horizontal line  $(1 + \rho) = h_2$  before  $y = h_1$ .

#### 4.2 Liquidity under-supply

As we saw above, for low enough  $\Pi$ , *B* firms cannot borrow all they would like, and it is no surprise that the equilibrium is not First Best efficient. However, we now show that equilibrium is not even Second Best efficient. The constraint on borrowing at time 1 induces optimizing *L* firms to under-invest in the liquid option at time 0. Had

they invested more in liquid capital, total output would have been higher, even if the borrowing constraint  $\Pi$  remained binding.

**Definition 2** A pair of decisions  $(\alpha_{CE}, y_{CE}, x_{CE})$  and prices  $\rho_{CE}$  is said to be *Constrained Efficient* if:

- (a) The choice  $\alpha_{CE}$  at  $t = 0$  maximizes total output at  $t = 2$ , assuming that at  $t = 1$  markets clear, i.e.,
- (b) Given  $\alpha_{CE}$  and  $\rho_{CE}$ ,  $L$  chooses  $y_{CE}$  and  $B$  chooses  $x_{CE}$  to maximize their own output.
- (c)  $x_{EQ} = y_{EQ}$ .

More precisely, we can think of a Constrained Efficient allocation  $(\alpha_{CE}, y_{CE}, x_{CE}, \rho_{CE})$ , as the one that solves the following maximization problem:

$$\left\{ \begin{array}{l} \max_{\alpha, y, x, \rho} \quad (1 - \alpha)H + (\alpha h_1 - y)h_2 + yR \\ \text{s.t.} \quad 0 \leq \alpha \leq 1 \\ y \text{ solves} \quad \left\{ \begin{array}{l} \max_z \quad (\alpha h_1 - z)h_2 + (1 + \rho_{CE})z \\ \text{s.t.} \quad 0 \leq z \leq \alpha h_1 \end{array} \right. \\ x \text{ solves} \quad \left\{ \begin{array}{l} \max_w \quad Rw - (1 + \rho_{CE})w \\ \text{s.t.} \quad 0 \leq (1 + \rho_{CE})w \leq \Pi \end{array} \right. \\ x = y \end{array} \right.$$

**Proposition 1** (Liquidity under-supply) *Suppose  $\Pi < H$  and  $h_1 h_2 < H < Rh_1$ . Then the equilibrium in this economy is not Constrained Efficient. In particular,  $\alpha_{EQ} < \alpha_{CE}$ , i.e., there is an under-supply of liquidity with respect to the Constrained Efficient allocation.*

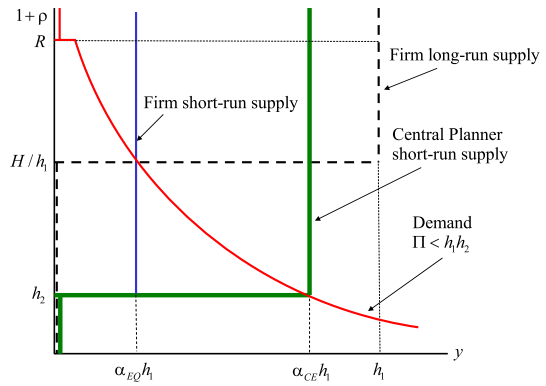
Having in mind the offsetting distortions principle, this result should not come as a big surprise. In fact, it is a particular case of that principle. Credit constraints,  $\Pi$ , play the role of demand distortions,  $\delta$ . As we saw, any distortion in *demand* can be understood as an inefficiency of *supply*, even though the demanders are completely different from the suppliers. Proposition 1 shows that this is also true for liquidity.

Consider first the case when  $\Pi < h_1 h_2$  as shown in Fig. 6. The only new element added to Fig. 5 is the filled curve to the right, which represents the short-run supply curve after constrained efficient choice  $\alpha_{CE}$ . As we can see in the figure,  $\alpha_{EQ} = \Pi/H < \alpha_{CE} = \Pi/h_1 h_2 < 1$ , so the equilibrium is not Constrained Efficient.

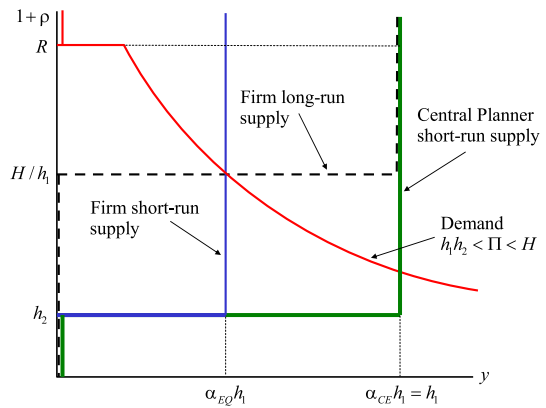
One can interpret the inefficiency as due to an externality. In equilibrium, lenders  $L$  at time  $t = 0$  forecast a depressed equilibrium interest rate  $1 + \rho$  and quantity at  $t = 1$ , coming from the intersection of their long-run supply curve and the constrained demand. Given this forecast, they optimally choose  $\alpha$  which determines their short-run supply. This curve intersects the demand at the same quantity and price  $H/h_1$ .

Suppose now, that firms  $L$  had chosen a bigger  $\alpha$  so that the short-run supply curve had been the one to the right. At this equilibrium, the interest rate is lower and the quantity borrowed is larger. Exactly this constitutes the externality, for the increase

**Fig. 6** Liquidity under-supply in the case  $\Pi < h_1 h_2$



**Fig. 7** Liquidity under-supply in the case  $h_1 h_2 < \Pi < H$



in the supply not only has the usual effect of lowering the price, but also loosens the borrowing constraint of firms  $B$ . These firms can make strictly positive profits from each extra unit they borrow due to the wedge between the equilibrium interest rate and the return  $R$ . Ex-ante, lenders  $L$  do not internalize this effect. From the social point of view, the supply of liquidity should be stimulated until the interest rate  $1 + \rho$  at  $t = 1$  falls to  $h_2$ . If the central planner knows he cannot restore demand for credit, then he must regard supply as in need of stimulation.

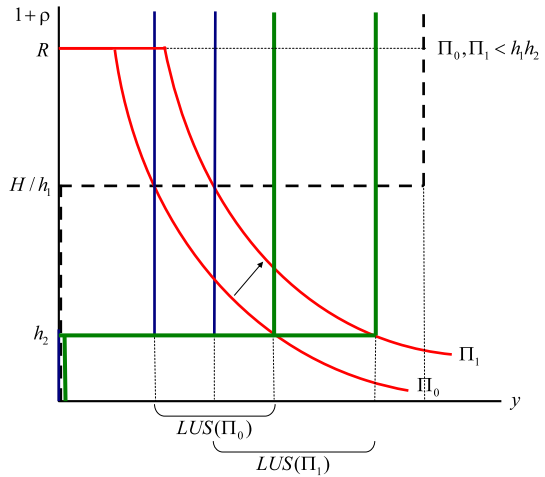
When  $h_1 h_2 \leq \Pi < H$ , a similar situation prevails, which we describe in Fig. 7. In this case  $\alpha_{EQ} = \Pi/H < \alpha_{CE} = 1$  and the central planner intervention attains the First Best quantity (even though not the First Best interest rate).

A last observation is that the result hinges only on the presence of credit constraints, in particular, it does not depend on the linear structure of the model.

### 4.3 Implementing the second best

We have seen that in equilibrium, optimizing agents will choose too little liquidity. A central planner could induce more liquid investments in several ways. Suppose however, that the only policy tool available to the planner is taxation. One method would

**Fig. 8** Increasing liquidity under-supply in the case  $\Pi < h_1 h_2$



be to tax the illiquid investment at time 0. The lack of government intervention at time 1 presumably would be explained by supposing the government could not distinguish between liquid investments, or because it could not modify  $h_2$  (for instance if  $h_2$  were the international interest rate).

**5 Non-monotonicity of the liquidity under-supply**

The central planner in our model is constrained. Since in the second period there is an outside option with return  $h_2$ , the central planner cannot modify the supply of loans below  $1 + \rho = h_2$ . This plays the role of  $p^*$  discussed before. Therefore, our model belongs to the Second Best world. As a result, we should expect a non-monotone relation between the liquidity under-supply, (in our previous terminology, the Second Best Inefficiency), and the credit constraint (the demand distortion) that creates it. This is exactly what Proposition 2 shows.

**Definition 3** For each  $\Pi$ , define the *Liquidity Under-Supply* as

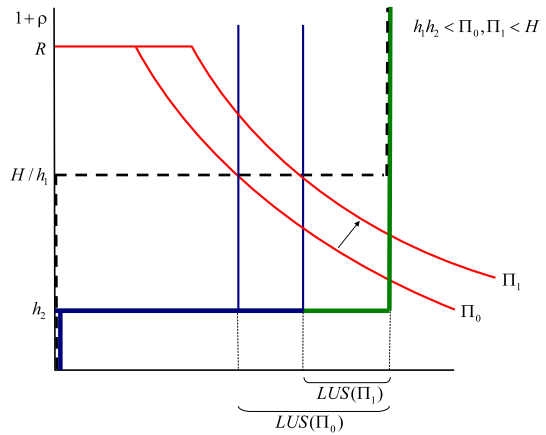
$$LUS(\Pi) = \alpha_{CE}(\Pi) - \alpha_{EQ}(\Pi)$$

**Proposition 2** (Non-Monotonicity of the liquidity under-supply) *Suppose  $\Pi < H$  and  $h_1 h_2 < H < Rh_1$ . Then the Liquidity Under-Supply is a non-monotone function of the credit constraint  $\Pi$ . In particular, the derivative satisfies*

$$\begin{aligned} LUS'(\Pi) &> 0 \quad \text{for all } \Pi < h_1 h_2 \\ LUS'(\Pi) &< 0 \quad \text{for all } \Pi \in (h_1 h_2, H). \end{aligned}$$

The non-monotonicity of the liquidity under-supply can be seen in Figs. 8 and 9. It is increasing in the “low  $\Pi$ ” zone while decreasing in the “high  $\Pi$ ” zone. In this simple framework, we can think of  $\Pi$  as the degree of financial development of the economy. Any financial innovation (expressed by an increase in  $\Pi$ ) at very low levels paradoxically makes government intervention (taxation) more necessary. As credit markets

**Fig. 9** Increasing liquidity under-supply in the case  $h_1 h_2 < \Pi < H$



begin to get more sophisticated, the distortions if markets work in a decentralized way become bigger. On the other hand, once the credit markets are sophisticated, as in the “high  $\Pi$ ” zone, credit market innovations lower the distortion and the costs of non-intervention become smaller. Although the model is too simple to really think about policy implications, we think this non-monotonicity property could have interesting implications for Emerging Markets economies.

Finally, note that the alternative investment opportunity  $h_2$  is a critical ingredient of the non-monotonicity. The essence of liquidity is flexibility, so the presence of such alternatives goes hand in hand with liquidity. Therefore, the non-monotonicity of the under-supply of liquidity is an inevitable consequence of its nature.

## 6 Endogenous credit constraints and the liquidity under-supply multiplier

### 6.1 The model

So far we have taken  $\Pi$  to be exogenous. Now we show that by introducing collateral explicitly into the model, the credit constraint can be taken to be endogenous. This is important, because once  $\Pi$  is endogenous, the demand for liquidity might become much more elastic, increasing the second best inefficiency and hence the liquidity under-supply.

Let us extend the model by introducing a perfectly durable and divisible good at time  $t = 1$ , called the collateral good, owned in its entirety of one unit by  $B$  firms. We will assume that there is no market at time 1 for this good. It gives no utility to any agent at time  $t = 1$ , but it is desired by  $B$  firms for consumption at time  $t = 2$ . The good is also useful because a  $B$  firm can use a proportion  $\beta \in (0, 1]$  of it as collateral for his borrowing at time  $t = 1$ . We suppose that  $B$  firms have no incentive to repay any money on their loans, but that the collateral can be seized by the lender and sold to make them whole. Since the good is perfectly durable, if its price at time  $t = 2$  is sure to be  $\Pi_2$ , then lenders will be willing to accept promises of  $\beta\Pi_2$  due at time

$t = 2$ , and therefore the credit constraint faced by the borrowers is

$$(1 + \rho)x \leq \beta \Pi_2$$

In this new context,  $\beta$  can be interpreted as the degree of financial development of the economy. This is a natural interpretation since financial markets become more sophisticated as the proportion of the durable goods in the economy that can be used as collateral increases.

Before we move on, we need to extend the definition of competitive equilibrium for this model.

**Definition 4** A *Collateral Equilibrium* allocation consists of lenders and borrowers decisions  $(\alpha^*, y^*, C_2^{L*}, c_2^{L*}, x^*, C_2^{B*}, c_2^{B*})$  and prices  $(\rho^*, \Pi_2^*)$  such that:

- (a)  $L$  firms choose liquidity  $\alpha^*$  in period 0, lending  $y^*$  in period 1 and collateral  $C_2^{L*}$  and consumption  $c_2^{L*}$  in period 2 such that they solve,

$$\left\{ \begin{array}{l} \max_{\alpha, y, C_2^L, c_2^L} U_L(c_2^L) \\ \text{s.t.} \quad 0 \leq \alpha \leq 1 \\ \quad \quad 0 \leq y \leq \alpha h_1 \\ \quad \quad c_2^L \leq (1 - \alpha)H + (\alpha h_1 - y)h_2 + y(1 + \rho^*) \end{array} \right.$$

- (b)  $B$  firms choose borrowing level  $x^*$  in period 1, collateral  $C_2^{B*}$  and consumption  $c_2^{L*}$  such that they solve

$$\left\{ \begin{array}{l} \max_{x, C_2^B, c_2^B} U_B(C_2^B, c_2^B) \\ \text{s.t.} \quad 0 \leq (1 + \rho^*)x \leq \beta \Pi_2^* \\ \quad \quad c_2^B + \Pi_2^* C_2^B \leq Rx - (1 + \rho^*)x + \Pi_2^* \end{array} \right.$$

- (c)  $x^* = y^*$ .
- (d)  $C_2^{L*} + C_2^{B*} = 1$ .

Since the supply of the collateral good is fixed at 1, any Pareto efficient allocation maximizes the output of the consumption good  $c_2$ . Thus again, we take  $c_2^L + c_2^B$  as our measure of welfare. Hence, as before, we define a constrained efficient choice of liquidity  $\alpha_{CE}$  at time  $t = 0$  as one that maximizes total output  $c_2^L + c_2^B$  in period 2 assuming that markets clear at  $t = 1$ .

Suppose first that the utility of consumption of the output  $c_2^B$  and the collateral  $C_2^B$  by  $B$  firms at time  $t = 2$  is

$$U_B(C_2^B, c_2^B) = c_2^B + \Pi_2 C_2^B$$

where  $\Pi_2$  is constant. It is evident that an equilibrium in this case is precisely the same as the one we computed earlier in Sect. 4. Central planner interventions could take many forms, but we concentrate on the taxation of illiquid investments. Starting

from an equilibrium in which  $1 + \rho = H/h_1$ , and  $0 < \alpha < 1$ , it is easy to see that a sufficiently small  $\varepsilon$  tax on illiquid investments would lower equilibrium  $1 + \rho$  by the same  $\varepsilon/h_1$ . But once we know equilibrium  $1 + \rho$ , we can compute the equilibrium borrowing  $x$  of firm  $B$  very easily. From the credit constraint we get that

$$\frac{dx}{d\rho} = -\frac{\Pi_2\beta}{(1 + \rho)^2}$$

A change in the interest rate has only a *direct effect* on the demand for loans. The situation is different when  $\Pi_2$  becomes an endogenous variable as we show next. We now turn to non-linear utility  $U_B$ .

Let us assume that  $B$  treats  $C_2^B$  and  $c_2^B$  as complements. Accordingly, define  $B$ 's marginal rate of substitution of  $c_2$  for  $C_2$  by  $\Pi_2(c_2^B)$  and assume

$$\begin{aligned} \Pi_2(c_2^B) &\equiv \frac{\partial U_B(1, c_2^B)/\partial C_2^B}{\partial U(1, c_2^B)/\partial c_2^B} \\ \Pi_2'(c_2^B) &> 0 \end{aligned}$$

**Proposition 3** (Liquidity Under-Supply Multiplier) *Suppose  $h_1h_2 < H < Rh_1$ . If in equilibrium  $B$  firms borrow  $x < h_1$ , then the change in equilibrium borrowing occasioned by a change in  $(1 + \rho)$  has derivative*

$$\frac{dx}{d\rho} = -\eta \frac{\Pi_2\beta}{(1 + \rho)^2}$$

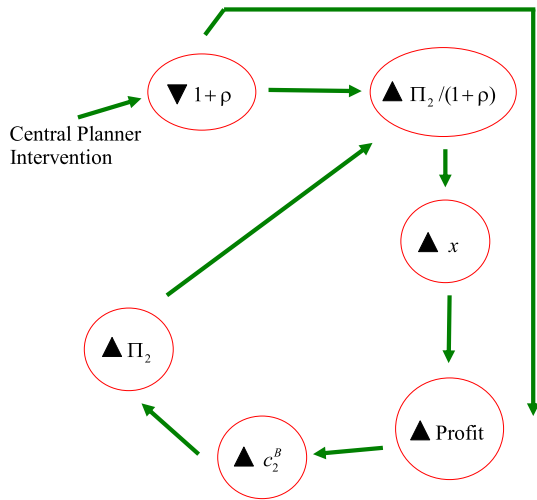
where,  $\eta = \{(1 + \beta\Pi_2')/(1 - [\beta\Pi_2'/(1 + \rho)])\}[(R - (1 + \rho))] > 1$ , is the *Liquidity Under-Supply Multiplier*.

Lower  $\rho$ , after intervention, enables the  $B$  firms to borrow more, even if  $\Pi_2$  does not change. This is the direct effect we already saw with  $\Pi_2$  fixed. Lower  $\rho$  also increases the profit of  $B$  firms, even if they borrowed the same amount. Borrowing more further increases their profit, since on the margin  $R > (1 + \rho)$ . Increasing the profit of the  $B$  firms boosts their consumption of  $c_2^B$  (the consumption of  $C_2^B$  cannot increase, since supply is fixed at 1). By assumption, this increases the relative marginal utility of  $C_2^B$ , thus increasing  $\Pi_2$  in the future. But lenders can forecast this future increment on  $\Pi_2$ , which in turn raises the ability of  $B$  firms to borrow, causing a big multiplier effect. This is the *indirect effect*. We can see this in Fig. 10.

Our amplification mechanism is similar in spirit to [Kiyotaki and Moore \(1997\)](#). In their work a constrained demand reduces the productivity of assets, causing the fall in asset prices and further constraining demand. Our paper pushes this mechanism a step further, noticing that the amplification reaches the supply side as well, since it makes the under-supply even bigger.



**Fig. 10** The liquidity under-supply multiplier



6.2 An example

In particular let us assume that  $U_L(C_2^L, c_2^L) = c_2^L$  and that  $U_B(C_2^B, c_2^B) = \mu c_2^B - (1/2)(c_2^B)^2 + \lambda C_2^B$ .

The following discussion is in terms of the parameters  $\beta$  and  $\lambda$ . As we said before,  $\beta$  is a measure of financial development of the economy, since the extent to which durable goods can be used as collateral depends on the presence of institutions like courts that guarantee that function. On the other hand, how efficient a good is in its function as collateral depends also on its market value, which depends on price times quantity. Since without loss of generality we choose units such that the total quantity of collateral is 1, this aspect is represented in our model by  $\lambda$ , the borrowers' marginal utility of collateral. The two variables are of vital importance. For instance, houses will not be a reasonable collateral if they have a very low value regardless of the presence of an efficient judicial system. Valuable goods will not be regarded as good collateral without a court system capable of enforcing confiscation. The following three propositions study the effect of  $\beta$  and  $\lambda$  on the liquidity under-supply, its behavior as a function of  $\beta$  and the liquidity under supply multiplier.

**Proposition 4** [Liquidity under-supply (LUS)] *Suppose  $h_1 h_2 < H < R h_1$ ,  $\mu > R h_1 + 1$  and  $\lambda > 0$ . Then  $\exists \lambda_1 > 0$  such that:*

- (a)  $\forall \lambda < \lambda_1, LUS(\beta) > 0, \forall \beta \in (0, 1]$ .
- (b)  $\forall \lambda > \lambda_1, \exists \beta_1(\lambda) > 0$  such that  $LUS(\beta) > 0, \forall \beta < \beta_1(\lambda)$ .

In economies where the market value of collateral is low there will always be liquidity under-supply, no matter how high  $\beta$  is. If the government cannot change  $\beta$  or  $\lambda$ , it should, for example, tax illiquid investments. However, when the market value of collateral is high enough, there exists liquidity under supply provided the level of financial development is low.

**Proposition 5** (Non-monotonicity of LUS) *Suppose  $h_1 h_2 < H < R h_1$ ,  $\mu > R h_1 + 1$  and  $\lambda > 0$ . Then  $\exists \lambda_0 < \lambda_1$  such that:*

- (a)  $\forall \lambda > \lambda_0$ ,  $LUS(\beta)$  is a non-monotone function. This is,  $\exists \beta_0(\lambda) < \beta_1(\lambda)$  such that  $LUS'(\beta) > 0, \forall \beta < \beta_0(\lambda)$  and  $LUS'(\beta) \leq 0, \forall \beta > \beta_0(\lambda)$
- (b)  $\forall \lambda < \lambda_0$ ,  $LUS'(\beta) > 0, \forall \beta \in (0, 1]$ .

**Proposition 6** (Non-monotonicity of the LUS Multiplier) *Suppose  $h_1 h_2 < H < R h_1$ ,  $\mu > R h_1 + 1$  and  $\lambda > 0$ . Then*

- (a)  $\forall \lambda > \lambda_1$ ,  $\eta(\beta)$  is a non-monotone function, this is,  $\eta'(\beta) > 0, \forall \beta < \beta_1(\lambda)$  and  $\eta'(\beta) \leq 0, \forall \beta > \beta_1(\lambda)$ .
- (b)  $\forall \lambda < \lambda_1$ ,  $\eta'(\beta) > 0, \forall \beta \in (0, 1]$ .

Propositions 5 and 6 can be summarized as follows. When the market value of collateral is too low, any financial innovation expressed by an increase in  $\beta$ , regardless of how sophisticated the economy was to start with, will lead to a *bigger* liquidity under-supply. This inefficiency is even more dramatic, since the multiplier also gets bigger with the innovation. On the other hand, for higher levels of marginal utility of collateral, the behavior of the liquidity under-supply and the multiplier becomes non-monotone. For very low levels of financial development, any increase in  $\beta$  will make the under-supply and the multiplier bigger. However, for developed markets, any innovation will reduce both.

The implications of the example for emerging markets are potentially interesting. These economies are often characterized by durable (often non-tradable) goods with low market values (at least during big crises) or weak court systems that reduce the range of goods that can serve as collateral. It is difficult not to think about the financial innovation and simultaneous, dramatic, reduction of government participation in financial markets that took place in Latin American economies during the 1990s. Numerous liquidity crises occurred in these economies during that period.

Finally, to illustrate more clearly the example, we run a simulation of the model for parameter values of  $R = 3, H = 2, h_1 = 1, h_2 = 1$  and  $\mu = 7$ . It turns out that the cut values for  $\lambda$  are  $\lambda_0 = 6, \lambda_1 = 12$ . The following figures present the result of the simulation. Figure 11 shows the case for  $\lambda = 5$ , in which there is always liquidity under-supply, and both the liquidity under-supply and the multiplier are increasing.

Figure 12 corresponds to  $\lambda = 10$ , in which the liquidity under-supply is always positive but now becomes non-monotone as a function of  $\beta$ . The multiplier is still increasing.

Finally, Fig. 13 shows the case  $\lambda = 20$ , liquidity under-supply can be zero for high enough values of  $\beta$ , and both liquidity under-supply and the multiplier become non-monotone.

### 7 Liquidity under-supply in a stationary overlapping generations model

So far we have assumed a highly non-stationary environment in which the borrower only arrives on the scene when the lender is already middle-aged. One might wonder

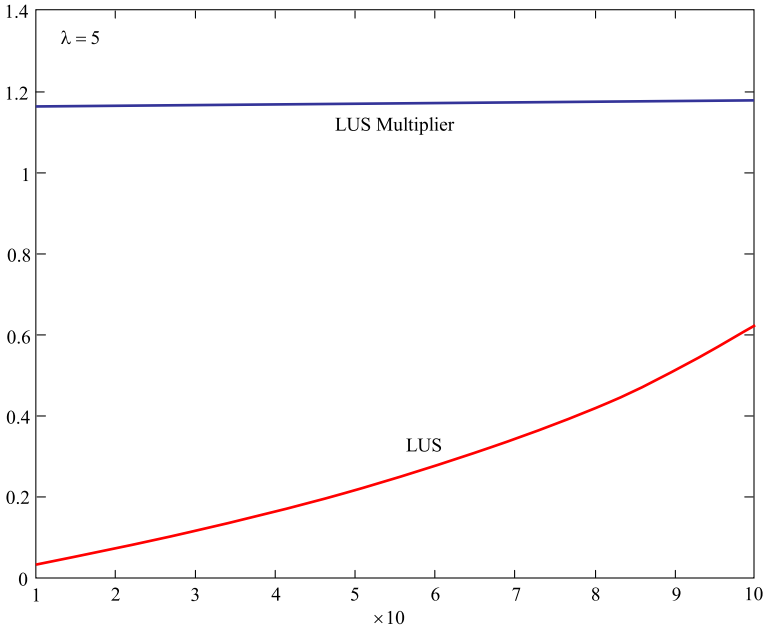


Fig. 11 Case  $\lambda = 5$

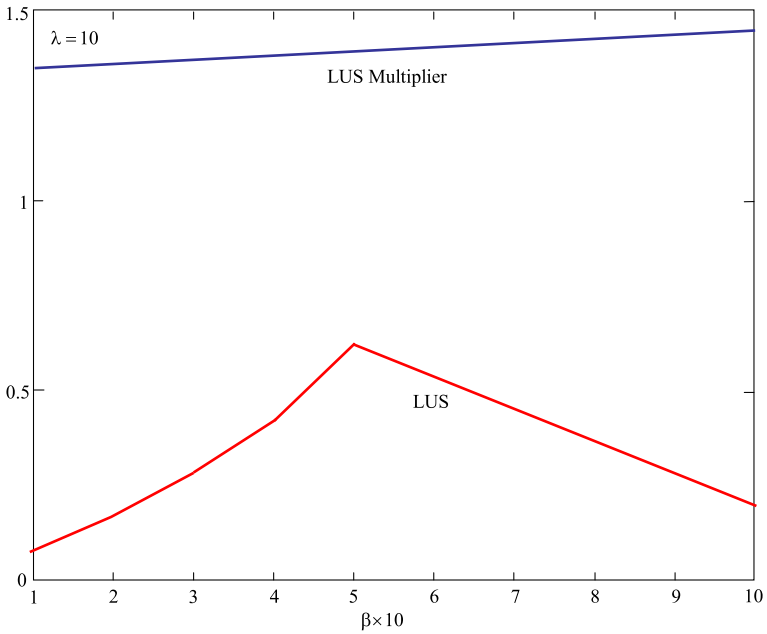


Fig. 12 Case  $\lambda = 10$

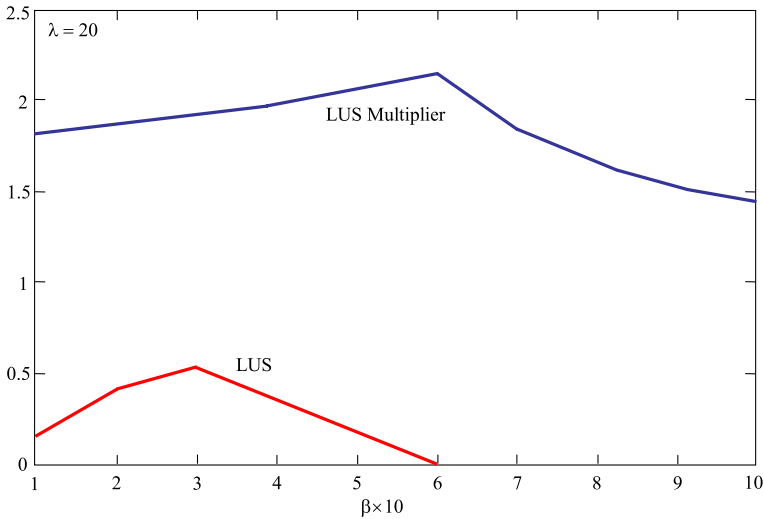


Fig. 13 Case  $\lambda = 20$

if our results remain intact if we assume a stationary environment. In fact, the under-supply becomes even more acute.

Suppose now that time extends indefinitely,  $t = \dots, -1, 0, 1, \dots$ . In every period  $t$  a new type  $B$  firm is born with the same technology we have already assumed. She can produce  $R$  units of output in period  $t + 1$  for every unit of input in period  $t$ . She has no endowment, and is constrained to borrow at most  $\Pi/(1 + \rho)$ , where  $\rho$  is the stationary one period interest rate on loans.

Each period a type  $L$  firm is born with an endowment of one unit of the good in its youth. The  $L$  firm lives three periods. As before, when young it can choose to put its wealth into an illiquid two-period investment that pays  $H$  per unit of input in period  $t + 2$ , or it can invest in a liquid one-period investment that pays  $h$  per unit input in period  $t + 1$ . In period  $t + 1$  it can take the receipts from its liquid investment from period  $t$  and invest that in another liquid investment, again paying  $h$  in period  $t + 2$  per unit invested in period  $t + 1$ . We suppose that  $h^2 < H < R^2$ .

The only difference is that now the  $L$  firm has another liquid option. It can lend in its youth to the contemporaneous young  $B$  firm, and again it can lend in its middle age to the next young  $B$  firm. Let  $\alpha_0$  ( $\alpha_1$ ) be the amount loaned at rate  $h$  when the firm is young (middle-aged) and  $y_0$  ( $y_1$ ) be the amount loaned at rate  $(1 + \rho)$  when the firm is young (middle-aged). Now the liquidity of the firm when young is  $\alpha_0 + y_0$ , and the firm liquidity is  $\alpha_0 h + y_0(1 + \rho)$  when middle-aged.

Consumption in old age is

$$c = (1 - \alpha_0 - y_0)H + y_1(1 + \rho) + \alpha_1 h$$

$$\alpha_1 + y_1 = \alpha_0 h + y_0(1 + \rho)$$

Equilibrium can take many forms, depending on the value of the exogenous  $\Pi$ . Suppose that  $\frac{\Pi}{\sqrt{H}(1+\sqrt{H})} < 1$ . Then it is easy to see that there is a stationary equilibrium in which

$$\begin{aligned}(1 + \rho) &= \sqrt{H} > h \\ \alpha_0 &= \alpha_1 = 0 \\ y_0 &= \frac{\Pi}{\sqrt{H}(1 + \sqrt{H})} \\ y_1 &= \frac{\Pi}{(1 + \sqrt{H})}\end{aligned}$$

The young lender puts  $1 - \frac{\Pi}{\sqrt{H}(1+\sqrt{H})}$  into the illiquid investment, and invests  $\frac{\Pi}{\sqrt{H}(1+\sqrt{H})}$  as a loan to the young  $B$  firm. At the same time the middle-aged  $L$  firm invests  $\frac{\Pi}{(1+\sqrt{H})}$  into the same  $B$  firm. The  $B$  firm is borrowing up to its limit of

$$\Pi/(1 + \rho) = \frac{\Pi}{\sqrt{H}} = \frac{\Pi}{\sqrt{H}(1 + \sqrt{H})} + \frac{\Pi}{(1 + \sqrt{H})}.$$

Equilibrium is constrained inefficient. Suppose the central planner is constrained not to touch  $h$  or  $\Pi$ . Nevertheless, he could boost output by taxing  $H$  at the rate  $t$  solving  $H(1 - t) = h^2$ . The young  $L$  firm would shift part of its wealth to the liquid investment, driving the equilibrium interest rate  $1 + \rho$  down to  $h$ . This would loosen the credit constraint allowing  $B$  firms to borrow more than in the decentralized equilibrium described above.

In this stationary framework the tax is doubly effective. When the young  $L$  firm shifts funds from the illiquid investment into the liquid loan to the  $B$  firm, it boosts output at once, since  $R > \sqrt{H}$ . And since this is a liquid investment, it also frees up more funds for the firm to loan when it is middle-aged, boosting output again.

The liquidity under-supply can be non-monotone in  $\Pi$ . As the reader may remember, that property relied only on the presence of an alternative investment opportunity  $h$  which is also present in this model. If we were to endogenize  $\Pi$ , as we did in Sect. 6, we would also get a liquidity under-supply multiplier.

## 8 Uncertainty

Uncertainty has played no role in our model. Indeed, it is not needed to generate under-supply or non-monotonicity. Introducing it does not change the qualitative features of the under-supply, but it can increase its magnitude.

Suppose now that the return  $R$  of the  $B$  firms is stochastic: with probability  $p$  it is  $R$ , as before, but with probability  $(1 - p)$  it is 0. The idea is that in normal states of nature, with probability  $1 - p$ , the  $B$  firms have no opportunity to invest at time  $t = 1$ . In extraordinary, perhaps crisis, situations they have a huge opportunity or need

to invest with borrowed money. The question is, will liquidity providers, rationally anticipating these events, provide for the right amount of liquidity at  $t = 0$ ?

The answer is no, for the reasons given without uncertainty. Now, in equilibrium  $H = (1 - p)h_1h_2 + ph_1(1 + \rho)$ , where  $\rho$  is the interest rate in the event  $R > 0$ . It is easy to see that if  $p$  is small, and  $H > h_1h_2$ , then  $\rho$  can be enormous, choking off almost all borrowing by the  $B$  firms just when they need the money the most. The under-supply can therefore be much more severe.

### 9 Appendix

#### Proof of Proposition 1

We prove the proposition by calculating the unique equilibrium in this economy and comparing it with the social planner solution. In Sect. 4 we already showed that the equilibrium in the case  $\Pi < H$  is given by  $\{(\alpha_{EQ}, y_{EQ}, x_{EQ}), (1 + \rho_{EQ})\} = \{(\Pi/H, \Pi h_1/H, \Pi h_1/H), H/h_1\}$ .

The planner chooses  $\alpha$  in order to maximize total output at  $t = 2$ . It is clear that in the case  $\Pi < h_1h_2$ , this is done for  $\alpha = \Pi/h_1h_2$ . Therefore the constrained efficient solution is  $\{(\alpha_{CE}, y_{CE}, x_{CE}), (1 + \rho_{CE})\} = \{(\Pi/h_1h_2, \Pi, \Pi), h_2\}$ . When  $\Pi \in [h_1h_2, H]$ , by the same type of reasoning, we have that  $\{(\alpha_{CE}, y_{CE}, x_{CE}), (1 + \rho_{CE})\} = \{(1, h_1, h_1), \Pi/h_1\}$ .

Now, just notice that the equilibrium allocation is different from the constrained efficient allocation, and in particular,  $\alpha_{EQ} = \Pi/h < \alpha_{CE} = \Pi/h_1h_2$  since  $h_1h_2 < H$  in the case  $\Pi < h_1h_2$ . In the case in which  $\Pi \in [h_1h_2, H]$  we have that  $\alpha_{EQ} = \Pi/H < \alpha_{CE} = 1$ .

#### Proof of Proposition 2

It is a straightforward calculation to see that  $LUS'(\Pi) = 1/h_1h_2 - 1/H > 0$  in the case  $\Pi < h_1h_2$ , while  $LUS'(\Pi) = -1/H < 0$  in the case  $\Pi \in [h_1h_2, H]$  using the allocations calculated in the proof of Proposition 1.

#### Proof of Proposition 3

In equilibrium when the borrowing constraint is binding we must have<sup>3</sup>  $c_2^B = Rx - (1 + \rho)x$  and  $x = \beta\Pi_2(c_2^B)/(1 + \rho)$ . Therefore  $x(\rho) = \beta\Pi_2(c_2^B)/(1 + \rho) = \beta\Pi_2(Rx(\rho) - (1 + \rho)x(\rho))/(1 + \rho)$ . Hence,  $dx = -[\beta\Pi_2/(1 + \rho)^2]d\rho + [\beta\Pi_2'/(1 + \rho)]\{[R - (1 + \rho)]dx - x d\rho\}$ . Rearranging terms and using  $x = \beta\Pi_2/(1 + \rho)$  gives,  $dx/d\rho = \{\beta\Pi_2(1 + \beta\Pi_2')/[1 - \beta\Pi_2'/(1 + \rho)]\}[R - (1 + \rho)] \cdot 1/(1 + \rho)^2$ . Denote by  $\eta = \{(1 + \beta\Pi_2')/[1 - \beta\Pi_2'/(1 + \rho)]\}[R - (1 + \rho)]$  the liquidity under-supply

<sup>3</sup> It is clear that in equilibrium  $c_2^B = 1$ , hence the expression for consumption below follows from the budget constraint.

Multiplier. Then, we have that  $dx/d\rho = -\eta[\beta\Pi_2/(1 + \rho)^2]$ . Notice that as long as  $\Pi'_2 > 0$  and  $R > (1 + \rho)$ , this is much more sensitive.

Proof of Proposition 4

1. Calculation of Allocations

**Solving for period  $t = 2$**  It is very clear that lenders and borrowers decisions are  $C_2^L = 0, c_2^L = (1 - \alpha)H + (\alpha h_1 - y)h_2 + (1 + \rho)y, C_2^B = 1, c_2^B = Rx - (1 + \rho)x$ .

**Solving for period  $t = 1$**  Lenders decisions are exactly the same in Proposition 1. In analyzing borrowers decisions there are two cases depending on whether the credit constraint is binding or not.

**(i) Credit constraint binding** In this case we have that

$$x = \begin{cases} 0, & 1 + \rho > R \\ \left[0, \frac{\Pi_2}{1+\rho}\right], & 1 + \rho = R \\ \frac{\Pi_2}{1+\rho}, & 1 + \rho < R \end{cases} \tag{4}$$

From the budget constraint  $c_2^B = Rx - (1 + \rho)x$ . From the problem's first-order conditions we have that  $\Pi_2 = \lambda/(\mu - c_2^B)$ . Combining these last three expressions we get that  $\Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ . Plugging this expression for  $\Pi_2$  into (4), we get the demand for loans. To solve for equilibrium we just solve Eqs. (4) and (3) to get:

$$1 + \rho = \begin{cases} h_2, & \alpha \geq \frac{\beta\Pi_2(h_2)}{h_1h_2} \\ (1 + \rho)^*, & \alpha \leq \frac{\beta\Pi_2(h_2)}{h_1h_2} \end{cases} \tag{5}$$

where  $(1 + \rho)^*$  solves the equation  $\alpha h_1 = \beta\Pi_2(1 + \rho)/(1 + \rho)$ .

**(ii) Credit constraint not binding** In this case, the demand for loans is

$$x = \begin{cases} 0, & 1 + \rho > R \\ (0, \infty), & 1 + \rho \leq R \end{cases} \tag{6}$$

Furthermore,  $c_2^B = 0$  and from the first-order conditions  $\Pi_2 = \lambda/\mu$ . Finally to solve for equilibrium solve equations (3) and (6) to get that

$$1 + \rho = R \tag{7}$$

**Solving for period  $t = 0$   
Competitive equilibrium**

**(i) Credit constraint binding** The long-run supply of lenders is given by (2). Taking into account (5) and (2) we have two cases:

In the first one  $1 + \rho = H/h_1$  and therefore  $\alpha = \beta\Pi_2(H/h_1)/H$ .

The equilibrium allocation is:

Lenders:  $\alpha = \beta\Pi_2(H/h_1)/H, y = \alpha h_1, C_2^L = 0, c_2^L = (1 - \alpha)H + y(1 + \rho)$

Borrowers:  $x = \alpha h_1, C_2^B = 1, c_2^B = Rx - (1 + \rho)x$

Prices:  $(1 + \rho) = H/h_1, \Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ .

In the second case, one of the two following conditions hold:

(I)  $\beta\Pi_2(H/h_1)/H > 1$ , (II)  $\mu^2 + 4\lambda\beta(1 - R/(H/h_1)) < 0$ .

In this case  $\alpha = 1$  and  $1 + \rho$  is given implicitly by the equation  $h_1 = \beta\Pi_2(1 + \rho)/(1 + \rho)$ . The equilibrium allocation is:

Lenders:  $\alpha = 1, y = \alpha h_1, C_2^L = 0, c_2^L = (1 - \alpha)H + y(1 + \rho)$

Borrowers:  $x = \alpha h_1, C_2^B = 1, c_2^B = Rx - (1 + \rho)x$

Prices:  $(1 + \rho)$  is given implicitly by the equation  $h_1 = \beta\Pi_2(1 + \rho)/(1 + \rho)$  and  $\Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ .

**(ii) Credit constraint not binding** Considering the long-run supply (2), we have that since  $(1 + \rho) = R > H/h_1$  then  $\alpha = 1$ . The equilibrium allocation is:

Lenders:  $\alpha = 1, y = h_1, C_2^L = 0, c_2^L = Rh_1$

Borrowers:  $x = h_1, C_2^B = 1, c_2^B = 0$

Prices:  $(1 + \rho) = R, \Pi_2 = \lambda/\mu$

**Central planner**

**(i) Credit constraint binding** The central planner chooses  $\alpha$  in order to maximize total output at  $t = 2$ , this is,  $(1 - \alpha)H + (\alpha h_1 - y)h_2 + Ry$ . Since  $Rh_1 > H$ , it is clear that he will choose  $\alpha$  so that the  $B$  firms can borrow all the way to the maximum, this time endogenously determined. Again we have two cases:

In the first case,  $1 + \rho = h_2$  and therefore  $\alpha = \beta\Pi_2(h_2)/h_1h_1$ .

The constrained efficient allocation is:

Lenders:  $\alpha = \beta\Pi_2(h_2)/h_1h_2, y = \alpha h_1, C_2^L = 0, c_2^L = (1 - \alpha)H + y(1 + \rho)$

Borrowers:  $x = \alpha h_1, C_2^B = 1, c_2^B = Rx - (1 + \rho)x$

Prices:  $(1 + \rho) = h_2, \Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ .

In the second case, one of the two following conditions hold:

(I')  $\beta\Pi_2(h_2)/h_1h_2 > 1$ , (II')  $\mu^2 + 4\lambda\beta(1 - R/(h_2)) < 0$

In this case  $\alpha = 1$  and  $1 + \rho$  is given implicitly by the equation  $h_1 = \beta\Pi_2(1 + \rho)/(1 + \rho)$ .

The constrained efficient allocation is:

Lenders:  $\alpha = 1, y = \alpha h_1, C_2^L = 0, c_2^L = (1 - \alpha)H + y(1 + \rho)$

Borrowers:  $x = \alpha h_1, C_2^B = 1, c_2^B = Rx - (1 + \rho)x$

Prices:  $(1 + \rho)$  is given implicitly by the equation  $h_1 = \beta\Pi_2(1 + \rho)/(1 + \rho)$  and  $\Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ .

**(ii) Credit constraint not binding** This case is exactly as analyzed in the case of decentralized equilibrium above.



2. Proof of (a) and (b)

Consider  $\Delta(\lambda, \beta, \rho) = \mu^2 + 4\lambda\beta(1 - R/(1 + \rho))$  and  $\alpha(\lambda, \beta, \rho) = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2(1 - R/(1 + \rho))H$ . Let  $\lambda'$  such that solve  $\Delta(\lambda, 1, H/h_1) = 0$  and let  $\lambda''$  such that solve  $\alpha(\lambda, 1, H/h_1) = 1$ . Define  $\lambda_1 = \min\{\lambda', \lambda''\}$ .

- (a) Given the definition of  $\lambda_1$ , the fact that  $\alpha$  is an increasing function of both  $\beta$  and  $\lambda$  and  $\Delta$  is a decreasing function of both arguments, implies that  $\alpha_{EQ}(\beta) < 1 \forall \beta \in (0, 1]$ . In this case

$$LUS(\beta) = \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/h_2)}}{2(1 - R/h_2)h_1h_2} - \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - Rh_1/H)}}{2(1 - Rh_1/H)H}$$

Since  $H > h_1h_2$ ,  $LUS(\beta) > 0 \forall \beta \in (0, 1]$ . Obviously this holds even in the case  $\alpha_{CE} = 1$ .

- (b) Given the definition of  $\lambda_1$  and the fact that  $\alpha$  is a continuous increasing function of both  $\beta$  and  $\lambda$  and  $\Delta$  is a continuous decreasing function of both arguments, implies for any  $\lambda > \lambda_1$ ,  $\exists \beta_1(\lambda) > 0$  such that  $\alpha_{EQ} = 1$  and therefore  $LUS(\beta) > 0, \forall \beta < \beta_1(\lambda)$ .

Proof of Proposition 5

Let  $\lambda^+$  such that solve  $\Delta(\lambda, 1, h_2) = 0$  and let  $\lambda^{++}$  such that solve  $\alpha(\lambda, 1, h_2) = 1$ . Define  $\lambda_0 = \min\{\lambda^+, \lambda^{++}\}$ . Clearly  $\lambda_0 < \lambda_1$  since  $H > h_1h_2$ .

- (a) Given the definition of  $\lambda_0$  by an analogous argument of (b) in the proof of Proposition 4,  $\forall \lambda > \lambda_0, \exists \beta_0(\lambda) < \beta_1(\lambda)$  such that  $\alpha_{CE} = 1$ . Therefore  $\forall \beta < \beta_0(\lambda)$ , we have that

$$LUS(\beta) = \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/h_2)}}{2(1 - R/h_2)h_1h_2} - \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - Rh_1/H)}}{2(1 - Rh_1/H)H}$$

and

$$LUS'(\beta) = \frac{\lambda(1 - R/h_2)}{(1 - R/h_2)h_1h_2\sqrt{\mu^2 + 4\lambda(1 - R/h_2)}} - \frac{\lambda(1 - Rh_1/H)}{(1 - Rh_1/H)H\sqrt{\mu^2 + 4\lambda(1 - Rh_1/H)}}$$

It is easy to see that since  $H > h_1h_2$ ,  $LUS'(\beta) > 0$ .

On the other hand,  $\forall \beta_1(\lambda) > \beta \geq \beta_0(\lambda)$ , we have that

$$LUS(\beta) = 1 - \frac{-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - Rh_1/H)}}{2(1 - R_1/H)H}$$

and therefore,

$$\text{LUS}'(\beta) = -\frac{\lambda(1 - Rh_1/H)}{(1 - R_1/H)H\sqrt{\mu^2 + 4\lambda(1 - Rh_1/H)}} < 0$$

- Finally, it is clear that for  $\beta \geq \beta_1(\lambda)$ ,  $\text{LUS}(\beta) = 0$ , and therefore  $\text{LUS}'(\beta) = 0$ .  
 (b) It follows from definition of  $\lambda_0$  and the argument in the previous point for the case of  $\beta < \beta_0(\lambda)$ .

**Proof of Proposition 6**

From the equilibrium and the formula of Proposition 3,

$$\eta = \frac{1 + \frac{\lambda}{(\mu - \beta(\frac{R}{1+\rho} - 1)\Pi_2)^2}}{1 - \frac{\lambda}{(\mu - \beta(\frac{R}{1+\rho} - 1)\Pi_2)^2} \frac{1}{1+\rho}(R - (1 + \rho))} > 1$$

*Proof of (a)*

**Case  $\beta < \beta_1$**

First we prove that  $\Pi_2(\beta)$  is an increasing function. In this case the interest rate is constant and equal to  $H/h_1$ , and  $\Pi_2 = [-\mu + \sqrt{\mu^2 + 4\lambda\beta a}]/2\beta a$ , where  $a = (1 - Rh_1/H)$ . After some algebra,

$\Pi'(\beta) = [\mu - (\mu^2 + 2\lambda\beta a)/\sqrt{\mu^2 + 4\lambda\beta a}]/2a\beta^2$ . Since the denominator is negative, we only need to check that the denominator is negative as well. Define  $z(\beta) = \mu - (\mu^2 + 2\lambda\beta a)/\sqrt{\mu^2 + 4\lambda\beta a}$ . Then,  $z(0) = 0$  and  $z'(\beta) = -[2\lambda a/\sqrt{\mu^2 + 4\lambda\beta a}(\mu^2 + 4\lambda\beta a)](2\lambda a\beta) < 0 \forall \beta \in (0, 1]$ . Therefore the numerator is negative as well and  $\Pi'(\beta) > 0$ .

Now we show that  $\eta'(\beta) > 0 \forall \beta < \beta_1$ . From definition  $\eta(\beta) = [1 + f(\beta)]/[1 - f(\beta)k]$ , where  $f(\beta) = \lambda/(\mu - \beta(Rh_1/H - 1)\Pi_2)^2$  and  $k = h_1/H(R - H/h_1) > 0$ . Since  $\Pi'(\beta) > 0$  it is clear that  $f'(\beta) > 0$  as well. Now,  $\eta'(\beta) = [f'(\beta) + k]/(1 - f(\beta)k)^2$ . Since  $f'(\beta) > 0$  and  $k > 0$ , we have that  $\eta'(\beta) > 0$  as we wanted to show.

**Case  $\beta > \beta_1$**

In this case the interest rate is a function of  $\beta$  as well. The first thing we show is that  $\rho'(\beta) > 0$ . For this, notice first that  $\Pi_2$  is now a function of two variables,  $\Pi_2(\beta, \rho) = [-\mu + \sqrt{\mu^2 + 4\lambda\beta(1 - R/(1 + \rho))}]/2\beta(1 - R/(1 + \rho))$ . Its partial derivatives are  $\partial\Pi_2/\partial\rho = [R/(1 + \rho)^2/[2\beta(1 - R/(1 + \rho))^2]w(\beta, b)$  and  $\partial\Pi_2/\partial\beta = [1/2\beta^2(1 - R/(1 + \rho))]w(\beta, b)$ , where  $w(\beta, b) = \mu - (\mu^2 + 2\lambda\beta b)/(\sqrt{\mu^2 + 4\lambda\beta b})$  and  $b(\rho) = (1 - R/(1 + \rho))$ . Hence, we have that  $b(H/h_1 - 1) = a$  and  $w(\beta, a) = z(\beta)$ . The same argument that shows that  $z(\beta) < 0$  extends to show  $w(\beta, b) < 0$  since in equilibrium  $b < 0$  and hence  $\partial\Pi_2/\partial\rho < 0$ ,  $\partial\Pi_2/\partial\beta > 0$ .

Define  $F(\beta, \rho) = \beta\Pi_2 - h_1(1 + \rho)$ . In equilibrium this is zero. Moreover,  $F_\rho = \beta\partial\Pi_2/\partial\rho - h_1 < 0$ , and hence different from zero. Therefore, by the Implicit

Function Theorem,  $\rho$  is locally a well defined function of  $\beta$  and its derivative is given by  $\rho'(\beta) = -F_\beta/F_\rho$ . Since  $F_\beta = \Pi_2 + \beta\partial\Pi_2/\partial\beta > 0$ , it is immediate that  $\rho'(\beta) > 0$ .

Now we show that  $\eta'(\beta) < 0$ . Using the fact that  $h_1 = \beta\Pi_2(\beta, \rho)/(1 + \rho)$  in equilibrium, it is true that  $\eta(\beta) = [1 + h(\beta)]/[1 - h(\beta)k(\beta)]$ , where  $h(\beta) = \lambda/(\mu - (R - (1 + \rho))h_1)^2$  and  $k(\beta) = (R/(1 + \rho) - 1)$ . Since  $\rho'(\beta) > 0$ ,  $h'(\beta) < 0$  and  $k'(\beta) < 0$ . Let  $\beta_1 < \beta_2$ . Then  $\eta(\beta_1) = [1 + h(\beta_1)]/[1 - h(\beta_1)k(\beta_1)] > [1 + h(\beta_2)]/[1 - h(\beta_2)k(\beta_2)] = \eta(\beta_2)$ .

*Proof of (b)*

This case follows immediately from the definition of  $\lambda_1$  and the proof above in the case  $\beta < \beta_1$ .

## References

- Caballero, R., Krishnamurthy, A.: International and domestic collateral constraints in a model of emerging market crises. Mimeo, Department of Economics, MIT (2001)
- Chang, R., Velasco, A.: Liquidity crises in emerging markets: theory and policy. In: Bernanke, B., Rotemberg, J (eds.) NBER Macro Annual. MIT, Cambridge (1999)
- Diamond, P.: Money in search equilibrium. *Econometrica* **52**, 1–21 (1986)
- Diamond, D., Rajan, R.: Liquidity shortages and banking crises. Mimeo, University of Chicago (2001)
- Geanakoplos, J.: Promises, promises. In: Arthur, W.B., Durlauf, S., Lane, D (eds.) The economy as an evolving complex system, vol. II, pp. 285–320. Addison-Wesley, Reading (1997)
- Geanakoplos, J.: Liquidity, default, and crashes. In: Endogenous Contracts. General Equilibrium. Advances in Economics and Econometrics: Theory and Applications, Eighth World Conference II, Econometric Society Monographs, pp. 170–205 (2003)
- Holmstrom, B., Tirole, J.: Private and public supply of liquidity. *J Polit Econ* **106**, 1–40 (1998)
- Jones, R., Ostroy, J.: Flexibility and uncertainty. *Rev Econ Stud* **164**, 13–32 (1984)
- Kiyotaki, N., Moore, J.: Credit cycles. *J Polit Econ* **105**, 211–248 (1997)
- Kiyotaki, N., Moore, J.: Inside money and liquidity. Mimeo, London School of Economics (2000)
- Lipsey, R., Lancaster, K.: The general theory of second best. *Rev Econ Stud* **24**, 11–32 (1956)
- Morris, S., Shin, H.: Liquidity black holes. Cowles Foundation Discussion Paper No. 1434 (2003)
- Shubik, M.: The theory of money and financial institutions, vol. I. MIT, Cambridge (1999)