LEVERAGE AND DEFAULT IN BINOMIAL ECONOMIES: A COMPLETE CHARACTERIZATION

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Our paper provides a complete characterization of leverage and default in binomial economies with financial assets serving as collateral. Our Binomial No-Default Theorem states that any equilibrium is equivalent (in real allocations and prices) to another equilibrium in which there is no default. Thus actual default is irrelevant, though the potential for default drives the equilibrium and limits borrowing. This result is valid with arbitrary preferences and endowments, contingent or noncontingent promises, many assets and consumption goods, production, and multiple periods. We also show that only no-default equilibria would be selected if there were the slightest cost of using collateral or handling default. Our Binomial Leverage Theorem shows that equilibrium Loan to Value (*LTV*) for noncontingent debt contracts is the ratio of the worst-case return of the asset to the riskless gross rate of interest. In binomial economies, leverage is determined by down risk and not by volatility.

KEYWORDS: Endogenous leverage, default, collateral equilibrium, financial asset, binomial economy, *LTV*, diluted leverage, down risk.

1. INTRODUCTION

THE RECENT FINANCIAL CRISIS might well be understood as the bottom of a leverage cycle in which leverage, asset prices, and investment crashed together. It was preceded by years in which asset prices, investment, and the amount of leverage in the financial system increased dramatically. What determines leverage in equilibrium? Do these levels involve default? Our paper provides a complete characterization of leverage and default in *binomial* economies with *financial assets* serving as collateral.

Binomial models are the simplest economies in which uncertainty matters. It is not surprising, therefore, that they have played a central role in finance, such as in Black–Scholes pricing. These models are tractable because the binomial assumption ensures that trading just a stock and bond (that can be shorted) is enough to effectively complete markets. When collateral is needed to back promises, the binomial hypothesis no longer leads to complete markets. Our characterization shows that the binomial model with collateral is still tractable, and thus becomes the simplest tool to study the effect of endogenous leverage on asset prices, consumption, and investment.

Our first result, the Binomial No-Default Theorem, states that in binomial economies with financial assets serving as collateral, any equilibrium is equivalent (in real allocations and prices) to another equilibrium of the same economy in which there is no default. Thus potential default can have a dramatic

effect on equilibrium, but actual default never does. The Binomial No-Default Theorem is valid in a very general context with arbitrary preferences and endowments, contingent and noncontingent promises, many assets, many consumption goods, multiple periods, production, and with pyramiding (in which collateralized promises can serve as collateral for further promises).

The Binomial No-Default Theorem does not say that equilibrium is unique, only that each equilibrium can be replaced by another equivalent equilibrium in which there is no default. In particular, it proves the existence of a no-default equilibrium. However, we show that among all equivalent equilibria, the equilibria which use the least amount of collateral never involve default. These collateral minimizing equilibria would naturally be selected if there were the slightest transactions cost in using collateral or handling default. In these equilibria, we prove that the scale of promises per unit of collateral is unambiguously determined simply by the payoffs of the underlying collateral, independent of preferences or other fundamentals of the economy. Agents will promise as much as they can while assuring their lenders that the collateral is enough to guarantee delivery.

Any equilibrium with default can trivially be reproduced by an equilibrium without default in a different economy in which the promises have been changed to mimic the actual deliveries in the original equilibrium. If the original set of promises was incomplete, this typically would mean creating a new promise outside the cone spanned by the original promises. The No-Default Theorem presented here states something stronger: the no-default equilibrium can be obtained in the original incomplete markets economy without adding any new promises. For instance, if only noncontingent debt contracts are allowed in the original economy, then the theorem shows that there is an equilibrium in which all agents choose to trade only the safe noncontingent debt contract.

The No-Default Theorem has a sort of Modigliani–Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. It shows that if we start from any equilibrium with default, we can move to an equivalent equilibrium with less leverage, in which nobody defaults. The theorem does not say that starting from an equilibrium with no default, one can construct another equivalent equilibrium with even less leverage. Typically one cannot. Modigliani–Miller fails more generally in our model simply because the issuer of the debt must hold the collateral backing it.

Our second result, the Binomial Leverage Theorem, provides a simple formula for the Loan to Value (LTV), the ratio between the amount borrowed and the value of the asset used as collateral. It shows that when promises are noncontingent, as they typically are for the bulk of collateralized loans, the LTV on each financial asset in any collateral minimizing equilibrium is given by

$$LTV = \frac{\text{worst case gross rate of return}}{\text{riskless gross rate of interest}}.$$

The Binomial Leverage Theorem shows that leverage is endogenously determined in equilibrium by the Value at Risk equals zero rule, often assumed in the literature. Though simple and easy to calculate, this formula provides interesting insights. First, it explains which assets are easiest to leverage: the assets whose future value has the least down risk can be leveraged the most. Second, it explains why changes in down risk can have such a big effect on equilibrium even if they hardly change expected payoffs: they change leverage. The theorem suggests that one reason leverage might have plummeted from 2006 to 2009 is because the worst-case return that lenders imagined got much worse.

Many papers have assumed a link between volatility and leverage. The Binomial Leverage Theorem indeed shows that as the world becomes more dangerous, leverage goes down. But volatility is not, in general, an appropriate measure of danger. We show that it is possible to give examples in which higher volatility assets are leveraged more (because their volatility comes from upside risk and not downside risk). In the binomial world, down risk, not volatility, determines leverage.

Leverage is genuinely endogenous in Collateral Equilibrium. Equilibrium determines a menu, called the Credit Surface, specifying for each *LTV* from 0% up to 100% the lowest interest rate at which the market will lend. Borrowers are free to choose whichever leverage they want. In the classical model of perfect competition, where full delivery on promises is guaranteed by assumption, the Credit Surface is flat: each agent believes she could borrow as much as she likes at the same interest rate. In Collateral Equilibrium, the Credit Surface is flat as long as the collateral is big enough to guarantee delivery for sure, but is upward sloping once *LTV* surpasses a threshold, not because of oligopoly or market power effects, and not because of asymmetric information or other cash flow problems, but rather because the market understands that the same collateral does not as reliably back a bigger promise. If agents choose on the flat part of the Credit Surface, we say that leverage is demand determined. But if agents are choosing on the strictly increasing part of the Credit Surface, then we say that leverage is supply determined, since the agents would like to borrow more at the equilibrium interest rates, but do not because lenders would raise the rate.

The Binomial Leverage Theorem states that we can always assume that all agents are choosing the same point on the Credit Surface, namely the kink between the flat and the increasing part of the Credit Surface. The choice of leverage turns out to be independent of agent preferences, but determined exclusively by shocks to anticipated asset returns. Agents can borrow more than our *LTV* formula specifies, but they all choose not to do so because they do not want to pay a higher interest rate. Although agents are free to choose any leverage, they still may *feel* leverage constrained.

If an agent is choosing an *LTV* on the flat segment of the Credit Surface, she would be indifferent to moving right to the kink without paying any higher

interest simply by increasing the *LTV* on a smaller amount of collateral. This observation is the easy part of the proof of the Binomial No-Default Theorem, that we do not need to consider *LTVs* to the left of the kink. Thus an agent who is choosing the *LTV* at the kink of the Credit Surface may or may not be collateral constrained. For this reason, it is important to keep in mind another notion of leverage that we call *diluted investor LTV*, namely the ratio of total borrowing by an agent to her total asset value (including identical assets not used as collateral).

All our results depend on two key assumptions. First, we only consider financial assets, that is, assets that do not give direct utility to their holders, and which yield dividends that are independent of who holds them. Second, we assume that the economy is binomial, and that all loans are for one period.[2] A date-event tree in which loans last for just one period and every state is succeeded by exactly two nodes suggests a world with very short maturity loans and no big jumps in asset values. Binomial models might thus be taken as good models of Repo markets, in which the assets do seem to be purely financial, and the loans are extremely short term, usually one day.

The No-Default Theorem shows that there is a tremendous difference between physical collateral that generates contemporaneous utility and backs long term promises, and financial collateral that gives utility only through dividends or other cash flows, and backs very short term promises. Our result might explain why there are some markets (like mortgages) in which defaults are to be expected, while in others (like Repos) margins are set so strictly that default is almost ruled out.[3] The No-Default Theorem implies that if we want to study consumption or production or asset price effects of actual default (as opposed to studying the effect of leverage), we must do so in models that either include nonfinancial assets (like houses or asymmetrically productive land) or that depart from binomial models.

Our results also show why Brownian motion economies are not appropriate models to study supply determined leverage, though they can still be quite useful in modeling collateral equilibrium. Imagine a multi-period binomial economy with a stochastically growing asset that can be used as collateral to back one period noncontingent promises. It is well known that by taking shorter and shorter time periods, binomial economies converge to Brownian motion economies. As the period grows shorter, the one period volatility goes to zero and the one period worst-case return approaches the riskless return. From the Binomial Leverage Theorem, we conclude that the *LTV* on the shorter and

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[2]We could also allow for a long term loan with one payment date, provided that all the states at that date could be partitioned into two events, on each of which the loan promise and the asset value are constant.

[3]Even in 2007–2009 during the worst financial crisis of the last 70 years, Repo defaults, including of the Bear Sterns hedge funds, seem to have totaled a few billion dollars out of the trillions of dollars of Repo loans.

shorter one period loans converges to 100%. Thus in the limit, investors with a tiny bit of money can buy all the assets they want without running into a collateral constraint.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 presents a static model of endogenous leverage and debt with one asset and proves the main results in this simple case. Section 4 presents the general model of endogenous leverage and proves the general theorems. Section 5 presents two examples to illustrate our theoretical results. More details can be found in the Supplemental Material (Fostel and Geanakoplos (2015)).

2. RELATED LITERATURE

Collateral General Equilibrium was introduced by Geanakoplos (1997) and Geanakoplos and Zame (2014). The modeling strategy in Collateral General Equilibrium is characterized by two main ideas: first, it focuses on security-based leverage, and second, it is based on the idea that default happens when the collateral is worth less than the promise.[4] Collateral equilibrium embodies the idea of payment enforceability problems without cash flow problems. The only incentive to deliver on a promise is that otherwise the collateral can be seized. On the other hand, it is assumed that the cash flow from the collateral is not subject to any moral hazard or adverse selection; in particular, it is not affected by the size of the promise.

In Geanakoplos (1997), each contract is defined by a promise and one unit of an asset required as collateral to back the promise. The definition of equilibrium requires that every contract will have a price, and hence an associated *LTV* (the price of the contract divided by the price of its collateral), at which every agent can buy or sell as many units as she wants. By selling any of the contracts, an investor is borrowing money and putting up collateral; by buying any of the contracts, the investor is lending money. The menu of contracts using the same collateral and their associated prices can always be recast as a menu of *LTV*s and associated interest rates (the reciprocal of price), and is what Geanakoplos (2014) and we here call the Credit Surface. The Credit Surface endogenizes leverage because any investor can borrow or lend at any *LTV* she wants; if there were an *LTV* at which a lender and borrower would both prefer to transact, they would do so.

Geanakoplos (1997) asserted that because collateral is scarce, even if all contracts are priced in equilibrium, only a few will be actively traded. This, however, does not give a practical recipe for computing equilibrium leverage. Geanakoplos (2003) introduced a binomial economy with a continuum of agents with different priors, in which every agent was risk neutral and did not discount the future, and in which the agents' subjective probability of the

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[4]Another approach would be to enforce promises through punishment as in Dubey, Geanakoplos, and Shubik (2005).

up state increased monotonically and continuously in the index of the agent. In that special environment, he informally proved a slightly stronger version of our Binomial No-Default Theorem (that equilibrium is also unique). Fostel and Geanakoplos (2012a) formally proved that theorem. The Binomial No-Default Theorem proved in this paper is more general in that it does not depend on the number of agents, or on continuity of preferences across agents, or on identical discount rates, or on risk neutrality, or on any assumption about endowments (e.g., it does not assume that each agent's endowments in terminal periods is spanned by the asset). It includes the case where there is a finite number of agent types, as well as the case where there is a continuum of heterogeneous agents. With our complete characterization, binomial models with financial assets become a tractable tool to study leverage and its effect on asset prices and the real economy (see, e.g., Geanakoplos (2010), Fostel and Geanakoplos (2008, 2012b, 2014, 2016), and Cao (2010)).

Brunnermeier and Sannikov (2013) followed the Collateral General Equilibrium tradition in a continuous-time model with Brownian motion. In their model, leverage is also endogenous. However, as we discussed, an implication of our Binomial Leverage Theorem is that collateral constraints are irrelevant in the continuous-time limit of binomial models. Hence, leverage in their model is not determined by the asset collateral capacities, but by the agents' preferences, as it would be in a model without collateral requirements. In our terminology, their leverage is a demand-determined leverage, where agents are all choosing on the flat part of the Credit Surface.

There are no general results that characterize leverage in the tradition of this paper outside the binomial economies with financial assets considered here. But other papers have already given examples in which the No-Default Theorem does not hold. Geanakoplos (1997) gave a binomial example with a nonfinancial asset (a house, from which agents derive utility), in which equilibrium leverage is high enough that there is default. Geanakoplos (2003) gave an example with a continuum of risk-neutral investors with different priors and three states of nature in which the only contract traded in equilibrium involves default. Simsek (2013) gave an example with two types of investors and a continuum of states of nature with equilibrium default. Araujo, Kubler, and Schommer (2012) provided a two-period example of an asset which is used as collateral in two different actively traded contracts when agents have utility over the asset. Fostel and Geanakoplos (2012a) provided an example with three periods and multiple contracts traded in equilibrium.

Enforcement of promises through collateral has been used in many models but generally without endogenizing leverage. Mendoza (2010) assumed a fixed *LTV*. Gromb and Vayanos (2002) and Garleanu and Pedersen (2011) assumed a *VAR* = 0 rule, even though their models are not binomial. Brunnermeier and Pedersen (2009) assumed a *VAR* rule. In all these papers, the Credit Surface is truncated at an arbitrary point.

In our model, cash flow problems arising from asymmetric information play no role. In particular, the cash flows generated by the collateral do not depend on the promise it backs, or on whether they are seized by the lender after default. This contrasts with a literature that endogenizes investor-based leverage using corporate finance techniques with asymmetric information. In Stiglitz and Weiss (1981), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Holmstrom and Tirole (1997), Adrian and Shin (2010), Acharya and Viswanathan (2011), and Acharya, Gale, and Yorulmazer (2011) the endogeneity of leverage relies on moral hazard problems between lenders and borrowers. In one version, lenders insist that the manager puts up a portion of the investment himself in order to maintain skin in the game; otherwise, he would act differently and the collateral cash flows would be compromised.[5] In the literature following Rothschild and Stiglitz (1976), the endogeneity of leverage relies on adverse selection and signaling arguments. An agent limits the amount she borrows because she does not want to reveal that her collateral is of bad quality.

Another strand of the corporate finance literature with adverse selection, including Myers and Majluf (1984), DeMarzo and Duffie (1999), and Dang, Gorton, and Holmström (2011), focuses on the problem of optimal security design, where the issuer is choosing between promises that are not collinear. They found that noncontingent debt becomes optimal since it is less informational sensitive than equity. Our focus is on the comparison between a noncontingent promise that always pays, and a bigger noncontingent promise that defaults in one state. We generalize this question to a comparison of promises along any arbitrary ray, showing that it is optimal to issue the maximal promise along the ray that does not default. We do not pursue the question of comparing contracts along different rays, but we note that our proof shows that, in binomial economies, the collateral minimizing equilibrium always uses the ray which is farthest from the asset payoff. In particular, if Arrow securities can be issued, they will be. This result already appears in the unpublished work of Geanakoplos and Zame. But it shows that dropping the hypothesis of asymmetric information can lead to different conclusions, since in our framework debt is inferior to extreme Arrow promises.

Asymmetric information is important in loan markets for which the borrower is also a manager who exercises control over, or has special information about, the value of the collateral. Our work is complementary, since it does not rely on asymmetric information. The recent crisis, moreover, was not centered in the corporate bond world, but in the mortgage securities market, where the buyer/borrower generally has no control or specialized knowledge over the cash flows of the collateral.

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[5]In Kiyotaki and Moore (1997), it was assumed that lenders who seize collateral cannot get all the fruit it has produced.

3. LEVERAGE AND DEFAULT IN A SIMPLE MODEL OF DEBT

We first prove our results in a simple binomial model with only two periods, one asset, and noncontingent debt contracts. The key ingredients and arguments can be most easily understood in this simple framework. In Section 4, we present the more general model and explain in detail how the results in Section 3 extend to that framework.

3.1. *Model*

3.1.1. *Time and Assets*

We begin with a simple two-period general equilibrium model, with time $t = 0, 1$. Uncertainty is represented by different states of nature $s \in S$ including a root $s = 0$. We denote the time of s by $t(s)$, so $t(0) = 0$ and $t(s) = 1$, $\forall s \in S_T$, the set of terminal nodes of S. Suppose there is a single perishable consumption good c and one asset Y which pays dividends d_s of the consumption good in each final state $s \in S_T$. We take the consumption good as numeraire and denote the price of Y at time 0 by p. In the case of a binomial two-period tree, $S_T = \{U, D\}$. Figure 1 depicts the asset payoff in this case.

We call the asset a *financial asset* because it gives no direct utility to investors, and pays the same dividends no matter who owns it. Shares of General Electric stocks are financial assets. Houses are not financial assets because they give utility to their owners. Neither is land if its output depends on who owns it and tills it.[6]
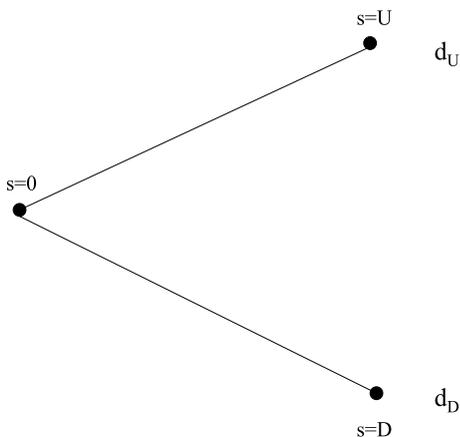

FIGURE 1.—Asset payoff.

[6]The literature on financial intermediation provides examples in which bonds (loans) owned by banks deliver different payoffs from bonds (loans) owned by less informed agents. These loans are not called financial assets in our model.

3.1.2. *Investors*

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, β_h, and subjective probabilities, γ_s^h, $s \in S_T$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+ \to R$, is differentiable, concave, and monotonic. The expected utility to agent h is[7]

$$(1) \qquad U^h = u^h(c_0) + \beta_h \sum_{s \in S_T} \gamma_s^h u^h(c_s).$$

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+$ in each state $s \in S$. Investor h's endowment of the only asset Y at time 0 is $y_{0*}^h \in R_+$. We assume that the consumption good is present in every state, $\sum_{h \in H} e_0^h > 0$, $\sum_{h \in H}(e_s^h + d_s y_{0*}^h) > 0$, $\forall s \in S_T$.

3.1.3. *Collateral and Debt Contracts*

The heart of our analysis involves collateral. Agents have access to an exogenously given set of financial contracts, each consisting of a promise and the collateral backing it. Agents cannot be coerced into honoring their promises except by seizing the collateral agreed upon by the contract. In this simple model, we suppose that agents can only make noncontingent promises, backed by the asset, that we call debt contracts.

A *debt contract* j promises $j > 0$ units of consumption good in each final state backed by one unit of asset Y serving as collateral. The terms of the contract are summarized by the ordered pair $(j \cdot \tilde{1}, 1)$. The first component, $j \cdot \tilde{1} \in R^{S_T}$ (the vector of j's with dimension equal to the number of final states), denotes the (noncontingent) promise. The second component, 1, denotes the one unit of the asset Y used as collateral. Let J be the set of all such available debt contracts.

The price π_j of contract j is determined in equilibrium. An investor can borrow π_j today by selling the debt contract j promising j tomorrow. Every contract j defines a gross rate of interest

$$(2) \qquad (1 + r_j) = j / \pi_j.$$

Notice that markets are incomplete. Agents cannot promise (d_U, d_D), which corresponds to "selling the asset short," nor can they promise either Arrow security. As we shall see in Section 4, our No-Default Theorem is all the stronger because it holds for any arbitrary set of contracts, including the complete markets case consisting of Arrow securities. The case of incomplete markets, such

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[7]All that matters for the results in this paper is that the utility $U^h : \mathbb{R}^S \to \mathbb{R}$ depends only on consumption (and not on portfolio holdings). The expected utility representation is done for familiarity. Our results will not depend on any specific type of agent heterogeneity either.

as we have here, is more surprising, since by defaulting an agent can effectively deliver (and with rational expectations, be paid for) a promise that is not available without default.

3.1.4. Leverage

Once the prices p and π_j are determined in equilibrium, the *Loan to Value* (*LTV*) associated to debt contract j is given by

$$(3) \qquad LTV_j = \frac{\pi_j}{p}.$$

The margin requirement m_j associated to debt contract j is $1 - LTV_j$, and the leverage associated to debt contract j is the inverse of the margin, $1/m_j$.

Let φ_j be the number of contracts j traded at time 0. There is no sign constraint on φ_j; a positive (negative) φ_j indicates the agent is selling (buying) $|\varphi_j|$ contracts j or borrowing (lending) $|\varphi_j|\pi_j$. We define the *average loan to value*, LTV^Y for asset Y, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium, and the *diluted average loan to value*, $DLTV^Y$ (which includes assets with no leverage), by

$$(4) \qquad LTV^Y = \frac{\displaystyle\sum_h \sum_j \max\left(0, \varphi_j^h\right)\pi_j}{\displaystyle\sum_h \sum_j \max\left(0, \varphi_j^h\right)p} \geq \frac{\displaystyle\sum_h \sum_j \max\left(0, \varphi_j^h\right)\pi_j}{\displaystyle\sum_h y_{0*}^h p} = DLTV^Y.$$

Finally, *leverage for investor h*, LTV^h, and the *diluted leverage for investor h*, $DLTV^h$, are defined analogously by

$$(5) \qquad LTV^h = \frac{\displaystyle\sum_j \max\left(0, \varphi_j^h\right)\pi_j}{\displaystyle\sum_j \max\left(0, \varphi_j^h\right)p} \geq \frac{\displaystyle\sum_j \max\left(0, \varphi_j^h\right)\pi_j}{y^h p} = DLTV^h.$$

3.1.5. Default and Delivery

We assume loans are nonrecourse, so the maximum borrowers can lose is their collateral if they do not honor their promise: the actual delivery of debt contract j in state $s \in S_T$ is

$$(6) \qquad \delta_s(j) = \min\{j, d_s\}.$$

Let $\overline{d} = \max\{d_s : s \in S_T\}$ and $\underline{d} = \min\{d_s : s \in S_T\}$. Observe that all the functions $\delta_s(j)$ are weakly increasing in j, and at least one of the functions is strictly

increasing for $0 \leq j \leq \overline{d}$. Therefore the equilibrium π_j must be strictly increasing for $0 \leq j \leq \overline{d}$. [8]

If the promise is small enough that $j \leq \underline{d}$, then the contract will not default, and in this case its price defines a gross riskless rate of interest $1 + r_j = 1 + r$.

3.1.6. *Budget Set*

Given the asset and debt contract prices $(p, (\pi_j)_{j \in J})$, each agent $h \in H$ chooses consumption, c_0, asset holding, y, and debt contract trades, φ_j, at time 0, and consumption, c_s, in each state $s \in S_T$, to maximize utility (1) subject to the budget set defined by

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\begin{aligned}
B^h(p, \pi) = \Big\{ & (c, y, \varphi) \in R_+^S \times R_+ \times R^J : \\
& (c_0 - e_0^h) + p(y - y_{0*}^h) \leq \sum_{j \in J} \varphi_j \pi_j \\
& (c_s - e_s^h) \leq y d_s - \sum_{j \in J} \varphi_j \min(j, d_s), \forall s \in S_T \\
& \sum_{j \in J} \max(0, \varphi_j) \leq y \Big\}.
\end{aligned}
$$

At time 0, expenditures on consumption and the asset, net of endowments, must be financed by money borrowed using the asset as collateral. In the final period, at each state s, consumption net of endowments can be at most equal to the dividend payment minus debt repayment. Finally, those agents who borrow must hold the required collateral at time 0. Notice that even with as many independent contracts as there are terminal states, equilibrium might still be different from Arrow–Debreu. Agents cannot willy-nilly combine these contracts to sell Arrow securities because they need to post collateral.[9]

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[8]Nobody would pay more for a contract that delivers less. If nobody is buying or selling a contract, there can be some ambiguity about its price. But we can always find an equilibrium in which π_j is indeed strictly increasing. In the binomial case this follows from our Binomial State Pricing Corollary in Section 3.2.

[9]Notice that we are assuming that short selling of assets is not possible. This is in keeping with the hypothesis of this paper that agents need to post collateral in order to back promises. The short sale of asset Y could be modeled as the sale of a contingent promise of (d_U, d_D) collateralized by some asset. In this section, we do not allow for contingent promises and the only asset is Y itself. In Section 4, we do introduce contingent promises and multiple assets, so short selling is indeed feasible in our Collateral Equilibrium. In Fostel and Geanakoplos (2012b), we investigated the significance of short selling and CDS for asset pricing.

3.1.7. *Collateral Equilibrium*

A *Collateral Equilibrium* is a vector consisting of an asset price, debt contract prices, individual consumptions, asset holdings, and contract trades $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H}) \in (R_+ \times R_+^J) \times (R_+^S \times R_+ \times R^J)^H$ such that

1. $\sum_{h \in H} (c_0^h - e_0^h) = 0$.
2. $\sum_{h \in H} (c_s^h - e_s^h) = \sum_{h \in H} y^h d_s, \forall s \in S_T$.
3. $\sum_{h \in H} (y^h - y_{0*}^h) = 0$.
4. $\sum_{h \in H} \varphi_j^h = 0, \forall j \in J$.
5. $(c^h, y^h, \varphi^h) \in B^h(p, \pi), \forall h$.
6. $(c, y, \varphi) \in B^h(p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h$.

In equilibrium, markets for the consumption good clear in all states. Markets for the asset and debt contracts clear at time 0, and agents optimize their utility in their budget sets. As shown by Geanakoplos and Zame (2014), equilibrium in this model always exists under the assumptions we have made so far.[10]

3.1.8. *The Credit Surface*

Given the asset and debt contract prices $(p, (\pi_j)_{j \in J})$, each agent $h \in H$ faces a menu of co-existing debt contracts $j \in J$. As we have just seen, π_j must be strictly increasing for $0 \leq j \leq \overline{d}$, hence the loan to value $LTV_j = \frac{\pi_j}{p}$ is strictly increasing in j (given the asset price p). Thus, LTV uniquely determines the contract j, and hence the gross interest rate $1 + r_j$.

We define the *Credit Surface* as this relationship between LTV and the gross rate of interest. Borrowers can choose any contract on the Credit Surface, provided they put up the collateral. Figure 2 shows the Credit Surface.

Let j^* be the contract that promises $\underline{d}$ in every $s \in S_T$. Point A in Figure 2 corresponds to the leverage and riskless interest rate, (LTV_{j^*}, r), of contract j^*. Point B in Figure 2 corresponds to leverage and interest rate, $(LTV_{j^{**}}, r_{j^{**}})$, of contract j^{**} promising $\overline{d}$ in every $s \in S_T$. Since j^{**} delivers $\delta_s(j^{**}) = d_s$ for all $s \in S_T$, in binomial economies it must be that $\pi_{j^{**}} = p$, and hence $LTV_{j^{**}} = 100\%$.[11] A critical property of the Credit Surface is that the gross interest rate

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[10]Our notation suggests that the set H of agents needs to be finite. But we could consider $H = [0, 1]$ as a continuum of distinct agents, without any conceptual changes to our model. In that case, replace the summation $\sum$ over agents by an integral over agents, and all the optimization conditions as holding with Lebesgue measure 1. Our theorems also hold for the continuum case. Examples of collateral equilibrium with a continuum of agents can be found in Geanakoplos (2003, 2010) and Fostel and Geanakoplos (2012a, 2012b).

[11]We shall show that in binomial economies all the debt on a given collateral, and the collateral itself, can be priced via state by state discounting of their payoffs. This is not true when $\#S_T > 2$. In the general case, all we know is that $p \geq \pi_{j^{**}}$.
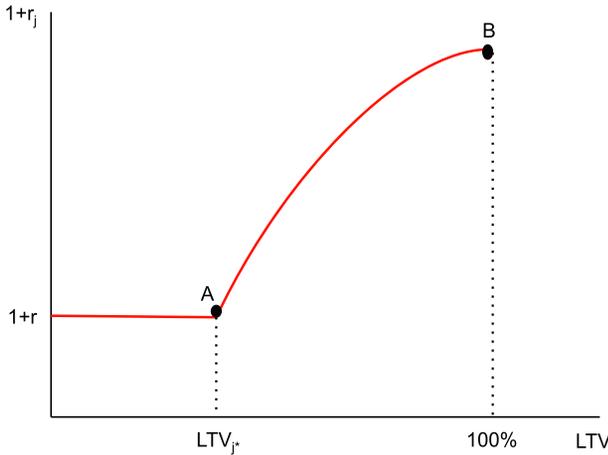
FIGURE 2.—The Credit Surface.

is strictly increasing in *LTV* between points *A* and *B*. The reason is that if $\underline{d} \leq j < j' \leq \overline{d}$, then $\frac{\delta_s(j')}{\delta_s(j)} \leq \frac{j'}{j}$ with a strict inequality for at least one state $s \in S_T$. Hence, in equilibrium $1 + r_{j'} = j'/\pi_{j'} > j/\pi_j = 1 + r_j$.[12]

In the budget set of the classical competitive model, borrowers can take as big a loan as they want at the riskless equilibrium interest rate *r*. In the collateral budget set, borrowers are forced to pay a higher interest rate for bigger loans (unless they put up more collateral). Thus they may choose to borrow π_j at an interest rate r_j, even though they would like to borrow more, because if they tried to borrow more on the same collateral, they would have to pay a higher interest rate.

During the Great Recession that started in 2008, many observers were perplexed that investors were borrowing so little money even when interest rates were close to zero. They suggested that investors could not find any projects with positive yields. The Credit Surface gives another explanation. In collateral equilibrium, it is perfectly possible that all equilibrium trades take place at a point like *A*, with a riskless rate of interest that might be close to zero. Investors may see plenty of profitable investments, but in order to undertake these projects they would have to increase their borrowing, which would increase the interest rate they would pay on all their borrowing. The higher interest rate that deters more borrowing might not be observable in market

[12]This argument presumes that both contracts are actively traded in equilibrium. If some of the contracts are not traded in equilibrium, we can always define prices so that this strict inequality holds. For an example of how to do this in a binomial economy, see the Binomial State Pricing Corollary in Section 3.

transactions; only the interest rates of actively traded contracts are observable.

3.2. *The Binomial No-Default Theorem*

3.2.1. *The Theorem*

Consider the case of a binomial tree in which $S_T = \{U, D\}$. As shown in Figure 1, asset Y pays d_U units of the consumption good in state $s = U$ and without loss of generality $0 < d_D < d_U$ in state $s = D$.[13] Default occurs in equilibrium if and only if some contract j with $j > d_D$ is traded. One might imagine that some agents value the asset much more than others, say because they attach very high probability γ_U^h to state U, or because they are more risk tolerant, or because they have very low endowments e_U^h in state U, or because they put a high value β_h on the future. These agents might be expected to want to borrow a lot, promising $j > d_D$ so as to get their hands on more money to buy more assets at time 0. Indeed, it is true that for $j > j^* = d_D$, agents can raise more money $\pi_j > \pi_{j_*}$ by selling contract j rather than j^*. We might expect every agent to choose a different point on the Credit Surface. Nonetheless, as the following result shows, we can assume without loss of generality that the only debt contract traded in equilibrium will be the max min contract j^*, on which there is no default.

BINOMIAL NO-DEFAULT THEOREM: *Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then given any equilibrium $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H})$, we can construct another equilibrium $((p, \pi), (c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same asset and contract prices and the same consumptions, in which j^* is the only debt contract traded, $\bar{\varphi}_j^h = 0$ if $j \neq j^*$. Hence equilibrium default can be taken to be zero.*

PROOF: The proof is organized in three steps.
1. Payoff Cone Lemma.
The portfolio of assets and contracts that any agent h holds in equilibrium delivers payoff vector (w_U^h, w_D^h) which lies in the cone positively spanned by $(d_U - j^*, 0)$ and (j^*, j^*). The Arrow U security payoff $(d_U - j^*, 0) = (d_U, d_D) - (j^*, j^*)$ can be obtained from buying the asset while simultaneously selling the max min debt contract.

Any portfolio payoff (w_U, w_D) is the sum of payoffs from individual holdings. The possible holdings include debt contracts $j > j^*$, $j = j^*$, $j < j^*$, the asset, and the asset bought on margin by selling some debt contract j. The debt contracts

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[13]Without loss of generality, $d_U \geq d_D$. If $d_D = 0$ or $d_D = d_U$, then the contracts are perfect substitutes for the asset, so there is no point in trading them. Sellers of the contracts could simply hold less of the asset and reduce their borrowing to zero while buyers of the contracts could buy the asset instead. So we might as well assume $0 < d_D < d_U$.
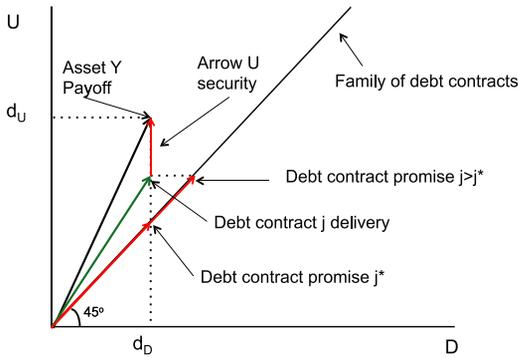
FIGURE 3.—Creating the Arrow U security.

and the asset all deliver at least as much in state U as in state D. So does the leveraged purchase of the asset. In fact, buying the asset on margin using any debt contract with $d_U > j \geq j^*$ is effectively a way of buying the Arrow U payoff $(d_U - j, 0)$. This can be seen in Figure 3.

In short, we have that the Arrow U security and the max min debt contract positively span all the feasible payoff space, as shown in Figure 4.

2. State Pricing Lemma.

There exist unique state prices a and *b such that if any agent h holds a portfolio delivering* (w_U^h, w_D^h), *the portfolio costs* $aw_U^h + bw_D^h$.

In step (i), we find state prices for two securities: the asset and the max min debt contract j^*. In steps (ii) and (iii), we use the Payoff Cone Lemma to show that the same state prices can be used to price any other debt contract $j \neq j^*$
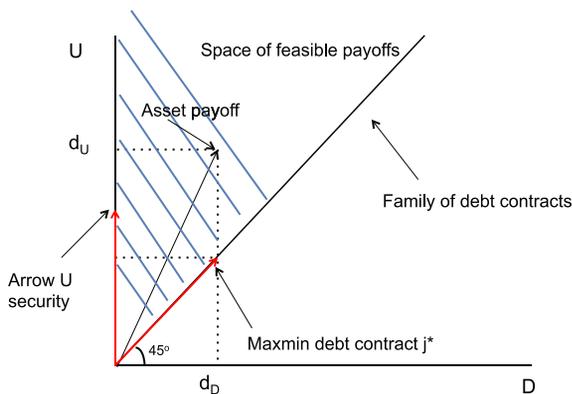

FIGURE 4.—Positive spanning.

that is traded in equilibrium. The cost of any portfolio is obtained as the sum of the costs of its constituent parts.

(i) *There exist unique a and b pricing the asset and the max min contract, that is, solving* $\pi_{j*} = aj^* + bj^*$ *and* $p = ad_U + bd_D$.

Since $d_U > d_D$, the two equations are linearly independent and therefore there exists a unique solution (a, b). It is easy to check that $a = \frac{p - \pi_{j*}}{d_U - j^*}$ and $b = \pi_{j*}/j^* - a$.

Notice that $a(d_U - j^*)$ is the price of buying $(d_U - j^*)$ Arrow U securities obtained by buying the asset Y and selling the max min contract j^*.

(ii) *Suppose debt contract j with* $j \neq j^* = d_D$ *is positively traded in equilibrium. Then* $\pi_j \leq a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\}$.

By the Positive Cone Lemma, the delivery of contract j is positively spanned by the Arrow U security $(d_U - j^*, 0)$ and the max min contract (j^*, j^*), both of which are priced by a and b. Hence buyers could obtain the same deliveries by buying a positive linear combination of the two, which would then be priced by a and b. This provides the upper-bound for π_j.

(iii) *Suppose debt contract j with* $j \neq j^* = d_D$ *is positively traded in equilibrium. Then* $\pi_j \geq a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\}$.

In case $j \leq j^* = d_D$, the contract fully delivers j in both states, proportionally to contract j^*. If its price were less than $\pi_{j*}(j/j^*) = aj + bj$, its sellers should have sold j/j^* units of j^* instead, which would have been feasible for them as it requires less collateral.

Consider the case $j > j^*$. Since any contract $j > d_U$ delivers exactly the same in both states as contract $j = d_U$, we can without loss of generality restrict attention to contracts j with $d_D < j \leq d_U$. Sellers of contract j have entered into a double trade, buying (or holding) the asset as collateral and selling contract j, at a net cost of $p - \pi_j$. Agents selling such a contract, while holding the required collateral, receive on net $d_U - j$ in state U, and nothing in state D. The key is that sellers of contract j are actually buyers of the Arrow U security. The cost is $p - \pi_j$ which, given step (i), is at most $a(d_U - j)$. Hence

$$p - \pi_j \leq a(d_U - j),$$
$$\pi_j \geq p - a(d_U - j),$$
$$\pi_j \geq ad_U + bd_D - a(d_U - j),$$
$$\pi_j \geq aj + bj^* = a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\}.$$

3. Construction of the new default-free equilibrium.
Define

$$\left(w_U^h, w_D^h\right) = y^h(d_U, d_D) - \sum_j \left(\min(j, d_U), \min(j, d_D)\right)\varphi_j^h,$$

$$\bar{y}^h = \frac{w_U^h - w_D^h}{d_U - d_D},$$

$$\bar{\varphi}_{j*}^h = [\bar{y}^h d_D - w_D^h]/j^* = \bar{y}^h - w_D^h/j^*.$$

If in the original equilibrium, y^h is replaced by $\bar{y}^h$ and φ_j^h is replaced by 0 for $j \neq j^*$ and by $\bar{\varphi}_{j*}^h$ for $j = j^*$, and all prices and other individual choices are left the same, then we still have an equilibrium.

(i) *Agents are maximizing in the new equilibrium.*

Note that $\bar{\varphi}_{j*}^h \leq \bar{y}^h$, so this portfolio choice satisfies the collateral constraint.

Using the above definitions, the net payoff in state D is the same as in the original equilibrium,

$$\bar{y}^h d_D - \bar{\varphi}_{j*}^h j^* = w_D^h,$$

and the same is also true for the net payoff in state U,

$$\bar{y}^h d_U - \bar{\varphi}_{j*}^h j^* = \bar{y}^h (d_U - d_D) + w_D^h = \left(w_U^h - w_D^h\right) + w_D^h = w_U^h.$$

Hence the portfolio choice $(\bar{y}^h, \bar{\varphi}_{j*}^h)$ gives the same payoff (w_U^h, w_D^h). From the previous lemmas, the newly constructed portfolio must have the same cost as well. Since Y is a financial asset, every agent is optimizing.

(ii) *Markets clear in the new equilibrium.*

Summing over individuals, we must get

$$\sum_h \bar{y}^h (d_U, d_D) - \sum_h \bar{\varphi}_{j*}^h (j^*, j^*)$$

$$= \sum_h \left(w_U^h, w_D^h\right)$$

$$= \sum_h y^h (d_U, d_D) - \sum_h \sum_j \varphi_j^h \left(\min(j, d_U), \min(j, d_D)\right)$$

$$= \sum_h y^h (d_U, d_D).$$

The first equality follows from step (i), the second from the definition of net payoffs in the original equilibrium, and the last equality follows from the fact that $\sum_h \varphi_j^h = 0$ in the original equilibrium for each contract j. Hence we have that

$$\sum_h \left(\bar{y}^h - y^h\right)(d_U, d_D) - \sum_h \bar{\varphi}_{j*}^h (j^*, j^*) = 0.$$

By the linear independence of the vectors (d_U, d_D) and (j^*, j^*), we deduce that

$$\sum_h \bar{y}^h = \sum_h y^h,$$

$$\sum_h \bar{\varphi}^h_{j^*} = 0.$$

Hence markets clear.   *Q.E.D.*

We now call attention to an interesting corollary of the proof just given. By modifying the equilibrium prices in the above construction for contracts that are not traded, we can bring them into line with the state prices a, b defined in the proof of the Binomial No-Default Theorem, without affecting equilibrium. More concretely, the following result holds.

BINOMIAL STATE PRICING COROLLARY: *Under the conditions of the Binomial No-Default Theorem, we may suppose that the new no-default equilibrium has the property that there exist unique state prices $a > 0$ and $b > 0$, such that $p = ad_U + bd_D$, and $\pi_j = a \cdot \min\{d_U, j\} + b \cdot \min\{d_D, j\}, \forall j \in J$.*

PROOF: The proof was nearly given in the proof of the Binomial No-Default Theorem.

It only remains to show that state prices are positive, that is, $a > 0$ and $b > 0$. First, $a > 0$; otherwise agents could buy more of Arrow U at a lower cost, violating agent optimization in equilibrium. We must also have $b > 0$; otherwise nobody would hold a portfolio paying $w_U > w_D$, since it would be better to buy w_U/j^* units of the contract j^*, which delivers the same w_U in state U and more $w_U > w_D$ in state D, and costs at most the same.

It is straightforward to show that if a previously untraded contract has its price adjusted into line with the state prices, then nothing is affected.   *Q.E.D.*

3.2.2. *Discussion*

The Binomial No-Default Theorem shows that in any static binomial model with a single financial asset serving as collateral, we can assume without loss of generality that the *only* debt contract actively traded is the max min debt contract, on which there is no default. In other words, we can assume that all agents choose the same point on the Credit Surface, given by point A in Figure 2, which corresponds to $(LTV_{j^*}, 1 + r)$. Thus *potential* default has a dramatic effect on equilibrium, but *actual* default does not.

The Binomial No-Default Theorem does not say that equilibrium is unique, only that each equilibrium can be replaced by another with the *same asset price and the same consumption* by each agent, in which there is no default.

The Binomial No-Default Theorem has a sort of Modigliani–Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. The theorem shows that if we start from any equilibrium, we can move to an equivalent equilibrium in which only max min debt is traded. If the original equilibrium had default, in the new equilibrium, leverage will be lower. Thus starting from a situation of default, the theorem does state that leverage can be lowered over a range until the point of no default, while leaving all investors indifferent. The theorem does *not* state that starting from a max min equilibrium, one can construct another equilibrium with still lower leverage, or even with higher leverage. Modigliani–Miller does not fully hold in our model because issuers of debt must hold collateral. In the traditional proof of Modigliani–Miller, given for example by Stiglitz (1969), when the firm issues less debt, buyers of its equity compensate by issuing debt themselves. But this argument relies on the fact that equity holders have enough collateral. In our model, if less debt is issued on a unit of collateral, then that collateral is wasted and there may not be enough other free collateral to back the new debt. In Section 5, we give an example with a unique equilibrium in which all the collateral is leveraged to the max min; if any collateral is wasted, total debt would have to go down, and Modigliani–Miller fails. But there is a simpler reason Modigliani–Miller can fail. In our model, the firm that issues the debt might not be able to sell off its equity because the equity is locked inside the collateral it must hold in order to issue the debt in the first place.

In the Binomial State Pricing Corollary, the state prices a, b are like Arrow prices. Their existence implies that there are no arbitrage possibilities in trading the asset and the contracts. Even agents who had infinite wealth and who were allowed to make promises without putting up any required collateral (but delivered as if they put up the collateral) could not find a trade that made money in some state without ever losing money. However, the equilibrium may not be an Arrow–Debreu equilibrium, even though the state prices are uniquely defined. We shall see an example with unique state prices but Pareto inferior consumptions (coming from the collateral constraints) in Section 5. Finally, the existence of state prices a and b allows us to see that the Credit Surface is not only increasing between points A and B, but also concave. Notice that, from equations (2) and (3) and the State Pricing Corollary, we have that $1 + r_j = \frac{LTV_j p - b d_D}{a LTV_j p}$.

Finally, let us provide some intuition to the proof of the No-Default Theorem. There are two key assumptions. First, we only consider financial assets, that is, assets that do not give direct utility at time 0 to their holders, and which yield dividends at time 1 that are independent from who holds them at time 0. Second, we assume that the tree is binary.

In the first step of the proof, the Payoff Cone Lemma shows that the max min promise plus the Arrow U security (obtained by buying the asset while selling the max min debt contract) positively spans the cone of all feasible portfolio payoffs. The assumption of two states is crucial. If there were three states, it

might be impossible for portfolio holders to reproduce their original net payoffs from a portfolio in which they can only hold the asset and buy or issue the max min debt.

In the second step of the proof, the State Pricing Lemma shows that any two portfolios that give the same payoffs in the two states must cost the same. One interesting feature of the proof is that it demonstrates the existence of state prices (that price the asset and all the debt contracts) even though short-selling is not allowed. In general, if an asset C has payoffs that are a positive combination of the payoffs from assets A and B, then the price of C cannot be above the positive combination of the prices of A and B. Otherwise, buyers or holders of C could improve by instead combining the purchase of A and B. This logic gives an *upper-bound* for prices of all traded instruments. On the other hand, the price of C could be less than the price of the positive combination of A and B (and just slightly more than the individual prices of A and B) because there may be no agent interested in buying it, and the sellers cannot split C into A and B. Nonetheless, we show that we can also get a *lower-bound* for the price of C. The reason is that in our model, sellers of the debt contract must own the collateral, and hence on *net* they are in fact buyers of something that lies in the positive cone. This gives us an upper-bound for the price of what they buy, and hence the missing lower-bound on what they sell. In short, the crucial argument in the proof is that sellers of contracts are actually buyers of something else that is in the payoff cone. As we will see later in Section 4, when there are multiple assets, or multiple kinds of loans on the same asset, the sellers of a bond in one family may not be purchasing something in the payoff cone of another family. Each family may require different state prices. That is why the No-Default Theorem holds more generally, but the State Pricing Corollary does not, so we can only show the existence of "local state prices."

In the third step of the proof, we use both lemmas to show that, in equilibrium, agents are indifferent to replacing their portfolio with another such that on each unit of collateral that they hold, they either leverage to the maximum amount without risk of default, or do not leverage at all. The idea is as follows. If, in the original equilibrium, investors leveraged their asset purchases *less* than the max min, they could always leverage some of their holdings to the max min, and the others not at all. This of course *reduces* the amount of asset they use for collateral. If, in the original equilibrium, investors were selling *more* debt than the max min, defaulting in state D, then they could again *reduce* their asset holdings and their debt sales to the max min level per unit of asset held, and still end up buying the same amount of the Arrow U security.[14] The reason they can reproduce their original net payoffs despite issuing fewer bonds per unit of collateral is that, on net, all contracts $j > j^*$ leave the collateral holders with some multiple of the Arrow U security. They simply must

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[14]If they continued to hold the same assets while reducing their debt to the max min per asset, then they would end up with more of the Arrow U security.

compensate by leveraging a different amount of collateral. By selling less debt per unit of collateral, they must spend more cash on each unit of the asset, so the reduction in asset holdings should not be surprising.[15] Once we see that the debt issuers can maintain the same net payoffs even if they issue less debt per unit of collateral, it is easy to see that their new behavior can be made part of a new equilibrium. Let the original buyers of the original risky bond buy instead all of the new max min debt plus all the asset that the original risky bond sellers no longer hold. By construction, the total holdings of the asset are unchanged, and the total holdings of debt are zero, as before. Furthermore, by construction, the sellers of the bond have the same portfolio payoff as before, so they are still optimizing. Since the total payoff is just equal to the dividends from the asset, and that is unchanged, the buyers of the bond must also end up with the same payoffs in the two states, so they are optimizing as well. The new portfolio may involve agents holding a new amount of the collateral asset, while getting the same payoff from their new portfolio of assets and contracts. Agents are indifferent to switching to the new portfolio because of the crucial assumption that the asset is a financial asset. If the collateral were housing or productive land, for example, the theorem would not necessarily hold.

3.3. *Equilibrium Refinements*

The Binomial No-Default Theorem states that every collateral equilibrium is equivalent to a "j^*-equilibrium" in which there is no default and j^* is the only contract traded. But the proof reveals more, namely that, in the j^*-equilibrium, agents use less of the asset as collateral, with one agent using strictly less, than in any other equivalent equilibrium. Thus, our theorem can be further sharpened if we add to the model some cost structure associated to either default or collateral use. More precisely, the following results hold.

NO-DEFAULT THEOREM REFINEMENT 1—Default Costs: *Suppose that $\varepsilon > 0$ units of the consumption good are lost after default. Then in every equilibrium, only debt contracts $j \leq j^*$ will be traded.*

PROOF: The proof follows immediately from the portfolio construction procedure in the Binomial No-Default Theorem, since, for all $j > j^*$, agents will incur unnecessary default costs.   *Q.E.D.*

The last theorem shows that if we add to the model a small cost to default, then our No-Default Theorem has more bite: now the equilibrium prediction always rules out default. Notice, however, that the equilibrium contracts may

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[15]To put it in other words, the debt on which they were defaulting provided deliveries that were similar to the asset (more in the state U than in state D), so when they sell less of these they must compensate by selling more of the asset and thus holding less.

not be unique, in the sense that agents may be leveraging less than in the max min level. The following result sharpens our theorem even more.

NO-DEFAULT THEOREM REFINEMENT 2—Collateral Costs: *Suppose that $\varepsilon > 0$ units of the consumption good are lost for every unit of asset used as collateral. Then in every equilibrium, only the debt contract j^* will be traded.*

PROOF: The proof follows immediately from the portfolio construction procedure in the Binomial No-Default Theorem, since it is always the case that if $j \neq j^*$ is traded in equilibrium, then some agent is using more collateral than would be required if he only sold j^*.   *Q.E.D.*

The last refinement shows that if we add to the model a small cost associated to collateral use, then j^* is the only contract traded in any equilibrium. This extra assumption is arguably realistic: examples of such costs are lawyer fees, intermediations fees, or even the more recent services provided by banks in the form of collateral transformation.

3.4. *Binomial Leverage Theorem*

3.4.1. *The Theorem*

The previous theorem gives an explicit formula for how many promises every unit of collateral will back in equilibrium, or equivalently, how much collateral will be needed to back each promise of one unit of consumption in the future. Leverage is usually defined in terms of a ratio of value to value, which also admits a simple formula. We now provide a characterization of endogenous leverage.

BINOMIAL LEVERAGE THEOREM: *Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then equilibrium LTV^Y can be taken equal to*

$$\frac{\pi_j^*}{p} = \frac{d_D/(1+r_{j^*})}{p} = \frac{d_D/p}{1+r_{j^*}} = \frac{\text{worst case rate of return}}{\text{riskless gross rate of interest}}.$$

PROOF: The proof follows directly from the Binomial No-Default Theorem. Since we can assume that, in equilibrium, the only contract traded is j^*, then

$$\frac{\pi_j^*}{p} = \frac{d_D/(1+r_{j^*})}{p} = \frac{d_D/p}{1+r_{j^*}}.$$ *Q.E.D.*

3.4.2. *Discussion*

The Binomial Leverage Theorem provides a very simple prediction about equilibrium leverage. According to the theorem, equilibrium LTV^Y for the

family of noncontingent debt contracts is the ratio of the worst-case return of the asset divided by the riskless gross rate of interest.

Equilibrium leverage depends on current and future asset prices, but is otherwise independent of the utilities or the endowments of the agents. The theorem shows that in static binomial models, leverage is endogenously determined in equilibrium by the Value at Risk equals zero rule, assumed by many other papers in the literature.

Though simple and easy to calculate, this formula provides interesting insights. First, it explains why changes in down risk can have such a big effect on equilibrium even if they hardly change expected payoffs: they change leverage. Hence, the theorem suggests that one reason leverage might have plummeted from 2006 to 2009 is because the worst-case return that lenders imagined got much worse: down risk increased. Second, the formula explains which assets are easier to leverage. In Section 4, we shall allow for leverage on multiple assets. The same formula for each asset shows that the asset with the least down risk can be leveraged the most.

Borrowers fall into two categories, constrained and unconstrained. Unconstrained borrowers are not designating all the assets they hold as collateral for their loans. In this case, they would not want to borrow any more at the going interest rates even if they did not need to put up collateral (but were still required, by threat of punishment, to deliver the same payoffs they would had they put up the collateral). Their demand for loans is then explained by conventional textbook considerations of risk and return. Constrained borrowers are posting all their assets as collateral. In this case of scarce collateral, borrowers would typically be willing to pay a higher interest rate to get hold of an extra dollar if they did not need to post the collateral.

At the aggregate, we can therefore have three regimes. First, when all the borrowers are unconstrained, we can say that the debt in the economy is determined by the demand for loans, and that the rate of interest clears the loan market without consideration of collateral or default. Second, when all the borrowers are constrained, we can say the debt in the economy is determined by the supply of credit, that is, by the maximum debt capacity of the assets (a level determined by the specter of default). And finally, the economy could be in an intermediate situation when some borrowers are constrained and some other borrowers are unconstrained. In this case, the debt level in the economy is determined by supply and demand.

Our theorem shows that in binomial models with financial assets, the equilibrium LTV^Y can be taken to be the same easy to compute number, no matter in which regime the economy is. The distinction between plentiful and scarce collateral all supporting loans at the same LTV^Y suggests that it is useful to keep track of a second kind of leverage that we introduced in Section 3.1.4 and called *diluted* leverage, $DLTV^Y$. Consider the following example: if the asset is worth $100 and its worst-case payoff determines a debt capacity of $80, then in equilibrium we can assume all debt loans written against this asset will have

LTV^Y equal to 80%. If an agent who owns the asset only wants to borrow $30, then she could just as well put up only three-eighths of the asset as collateral, since that would ensure there would be no default. The LTV^Y would then again be $30/$37.50 or 80%. Hence, it is useful to consider diluted $DLTV^Y$, namely the ratio of the loan amount to the total value of the asset, even if some of the asset is not used as collateral. The $DLTV^Y$ in this example is 30%, because the denominator includes the $62.50 of asset that was not used as collateral.[16]

If *diluted* leverage, $DLTV^Y$, is less than LTV^Y, we know that at least one agent is unconstrained. If $DLTV^Y$ and LTV^Y are close, then we can reasonably be sure that most agents are constrained.

Finally, it is often said that leverage should be related to volatility. Indeed, many papers have assumed this link. As we will discuss in Section 4, this link is not robust. What matter generally is down risk and not volatility.

4. A GENERAL BINOMIAL MODEL

In this section, we show that the irrelevance of actual default is a much more general phenomenon, as long as we maintain our two key assumptions: financial assets and binary payoffs. We formally allow for the following extensions.

Arbitrary one-period contracts: previously we assumed that the only possible contract promise was noncontingent debt. Now we allow for arbitrary promises (j_U, j_D), provided that the max min version of the promise $(\bar{\lambda} j_U, \bar{\lambda} j_D)$ where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(j_U, j_D) \leq (d_U, d_D)\}$ is also available.

Multiple simultaneous kinds of one-period contracts: not only can the promises be contingent, there can also be many different (noncollinear) types of promises co-existing. See Figure 5.

Multiple assets: we can allow for many different kinds of collateral at the same time, each one backing many (possibly) noncollinear promises.

Production and degrees of durability: the model already implicitly includes the storage technology for the asset. Now we allow the consumption goods to be durable, though their durability may be imperfect. We also allow for intra-period production. In fact, we allow for general production sets, provided that the collateral stays sequestered, and prevented from being used as an input.

Multiple goods: unlike our previous model, in each state of nature there will be more than one consumption good.

Multiple periods: we will extend our model to a dynamic model with an arbitrary (finite) number of periods, as long as the tree is binomial.

We also indicate that the Binomial No-Default Theorem can be further extended to pyramiding, in which contracts can serve as the collateral for further

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[16]Notice that in the proof of the Binomial No-Default Theorem, in moving from an old equilibrium in which only contracts $j < j^*$ are traded to the new max min equilibrium, diluted leverage stays the same, but leverage on the margined assets rises. In moving from an old equilibrium with default in which a contract $j > j^*$ is traded to the new max min equilibrium, diluted leverage strictly declines, and leverage on the margined assets also declines.
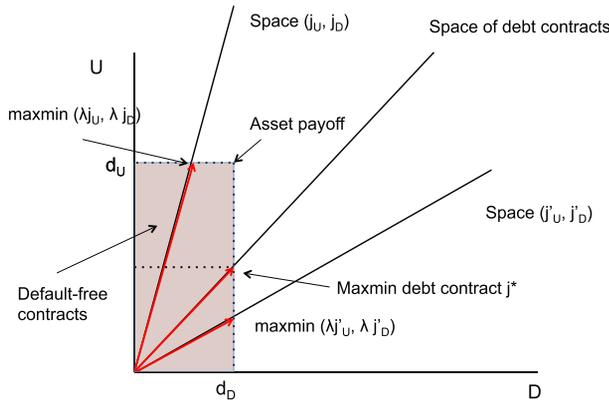
FIGURE 5.—Different types of contingent contracts.

contracts, but at the cost of too much notation to include here. It also extends to what we might call binomial assets of any duration in nonbinomial trees, as long as each (asset payoff, contract promise) pair takes on at most two values across all the states in which they pay.

4.1. *Model*

4.1.1. *Time and Assets*

Uncertainty is represented by the existence of different states of nature in a finite tree $s \in S$ including a root $s = 0$, and terminal nodes $s \in S_T$. We denote the time of s as $t(s)$, so $t(0) = 0$. Each state $s \neq 0$ has a unique immediate predecessor s^*, and each nonterminal node $s \in S \setminus S_T$ has a set $S(s)$ of immediate successors.

Suppose there are $L = \{1, \ldots, L\}$ consumption goods ℓ and $K = \{1, \ldots, K\}$ financial assets k which pay dividends $d_s^k \in \mathbb{R}_+^L$ of the consumption goods in each state $s \in S$. The dividends d_s^k are distributed at state s to the investors who owned the asset in state s^*.

Finally, $q_s \in R_+^L$ denotes the vector of consumption goods prices in state s, whereas $p_s \in R_+^K$ denotes the asset prices in state s.

4.1.2. *Investors*

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, β_h, and subjective probabilities γ_s^h denoting the probability of reaching state s from its predecessor s^*, for all $s \in S \setminus \{0\}$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+^L \to R$, is differentiable, concave, and

weakly monotonic (more of every good is strictly better). The expected utility to agent h is

$$(7) \qquad U^h = u^h(c_0) + \sum_{s \in S \setminus 0} \beta_h^{t(s)} \bar{\gamma}_s^h u^h(c_s),$$

where $\bar{\gamma}_s^h$ is the probability of reaching s from 0 (obtained by taking the product of γ_σ^h over all nodes σ on the path $(0, s]$ from 0 to s).

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+^L$ in each state $s \in S$. Investor h's endowment of the assets at the beginning of time 0 is $y_{0*}^h \in R_+^K$ (agents have initial endowment of assets only at the beginning). We assume that the consumption goods are all present, $\sum_{h \in H}(e_s^h + d_s y_{0*}^h) \gg 0$, $\forall s \in S$.

4.1.3. *Production*

We allow for durable consumption goods (inter-period production) and for intra-period production. For each $s \in S \setminus \{0\}$, let $F_s^h : \mathbb{R}_+^L \to \mathbb{R}_+^L$ be a concave inter-period production function connecting a vector of consumption goods at state s^* that h is consuming with the vector of consumption goods it becomes in state s. In contrast to consumption goods, it is assumed that all financial assets are perfectly durable from one period to the next, independent of who owns them. We keep production (of dividends) from financial assets separate from the production of commodities from commodities so that it is unambiguous what can be seized if an asset is put up as collateral. Implicitly, we also assume that production from assets is linear, again so that a lender can know what he will get if he is forced to seize the collateral, without having to observe how much of the asset the borrower is holding in his entire portfolio.

For each $s \in S$, let $Z_s^h \subset \mathbb{R}^{L+K}$ denote the set of feasible intra-period production for agent h in state s. Notice that assets and consumption goods can enter as inputs and outputs of the intra-period production process. Inputs appear as negative components of $z_i < 0$ of $z \in Z^h$, and outputs as positive components $z_i > 0$ of z.

4.1.4. *Collateral and Contracts*

Contract $j \in J$ is a contract that promises the consumption vector $j_{s'} \in R_+^L$ in each state s'. Each contract j defines its issue state $s(j)$, and the asset $k(j)$ used as collateral. We denote the set of contracts with issue state s backed by one unit of asset k by $J_s^k \subset J$. We consider one-period contracts, that is, each contract $j \in J_s^k$ promises deliveries only in the immediate successor states of s, that is, $j_{s'} = 0$ unless $s' \in S(s)$. Contracts are defined extensively by their promise in each successor state. Notice that this definition of contract allows for promises with different baskets of consumption goods in different states. Finally, $J_s = \bigcup_k J_s^k$ and $J = \bigcup_{s \in S \setminus S_T} J_s$.

The price of contract j in state $s(j)$ is π_j. An investor can borrow π_j at $s(j)$ by selling contract j, that is, by promising $j_{s'} \in R_+^L$ in each $s' \in S(s(j))$, provided he holds one unit of asset $k(j)$ as collateral.

4.1.5. *Leverage*

The *Loan-to-Value LTV_j* associated to contract j in state $s(j)$ is given by

$$(8) \qquad LTV_j = \frac{\pi_j}{p_{s(j)k(j)}}.$$

As before, the *margin m_j* associated to contract j in state $s(j)$ is $1 - LTV_j$. *Leverage* associated to contract j in state $s(j)$ is the inverse of the margin, $1/m_j$, and moves monotonically with LTV_j.

Finally, as in Section 3, we define the *average loan to value*, LTV for asset k in state s, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium that use asset k as collateral, $j \in J_s^k$ by all the agents $h \in H$, and the *diluted average loan to value*, $DLTV_s^k$ (which includes assets with no leverage), by

$$(9) \qquad LTV_s^k = \frac{\displaystyle\sum_h \sum_{j \in J_s^k} \max(0, \varphi_j^h)\pi_j}{\displaystyle\sum_h \sum_{j \in J_s^k} \max(0, \varphi_j^h)p_{sk}} \geq \frac{\displaystyle\sum_h \sum_{j \in J_s^k} \max(0, \varphi_j^h)\pi_j}{\displaystyle\sum_h y_{sk}^h p_{sk}} = DLTV_s^k.$$

Similarly, one can define *investor leverage, LTV_s^h*, and *diluted investor leverage, $DLTV_s^h$*, as

$$(10) \qquad LTV_s^h = \frac{\displaystyle\sum_k \sum_{j \in J_s^k} \max(0, \varphi_j^h)\pi_j}{\displaystyle\sum_k \sum_{j \in J_s^k} \max(0, \varphi_j^h)p_{sk}} \geq \frac{\displaystyle\sum_k \sum_{j \in J_s^k} \max(0, \varphi_j^h)\pi_j}{\displaystyle\sum_k y_{sk}^h p_{sk}} = DLTV_s^h.$$

4.1.6. *Default and Delivery*

As in Section 3, since the maximum borrowers can lose is their collateral if they do not honor their promise, the actual delivery of contract j in states $s' \in S(s(j))$ is

$$(11) \qquad \delta_{s'}(j) = \min\{q_{s'} \cdot j_{s'}, \, p_{s'k(j)} + q_{s'} \cdot d_{s'}^k\}.$$

4.1.7. *Budget Set*

Given consumption prices, asset prices, and contract prices (q, p, π), each agent $h \in H$ chooses intra-period production plans of goods and assets, $z = (z_c, z_y)$, consumption, c, asset holdings, y, and contract sales/purchases, φ, in

order to maximize utility (5) subject to the budget set defined by

$$B^h(q, p, \pi)$$

$$= \Big\{ (z_c, z_y, c, y, \varphi) \in R^{SL} \times R^{SK} \times R^{SL}_+ \times R^{SK}_+ \times (R^{J_s})_{s \in S \setminus S_T} :$$

$$q_s \cdot (c_s - e^h_s - F^h_s(c_{s*}) - z_{sc}) + p_s \cdot (y_s - y_{s*} - z_{sy})$$

$$\leq q_s \cdot \sum_{k \in K} d^k_s y_{s*k} + \sum_{j \in J_s} \varphi_j \pi_j - \sum_{k \in K} \sum_{j \in J^k_{s*}} \varphi_j \min\{q_s \cdot j_s, p_{sk} + q_s \cdot d^k_s\};$$

$$(z_{sc}, z_{sy}) \in Z^h_s; \forall s$$

$$\sum_{j \in J^k_s} \max(0, \varphi_j) \leq y^k_s, \forall k, \forall s \in S \setminus S_T \Big\}.$$

In each state s, expenditures on consumption minus endowments plus any produced consumption good (either from the previous period or produced in the current period), plus total expenditures on assets minus asset holdings carried over from previous periods and asset output from the intra-period technology, can be at most equal to total asset deliveries plus the money borrowed selling contracts, minus the payments due at s from contracts sold in the past. Intra-period production is feasible. Finally, those agents who borrow must hold the required collateral. We define $c_{0*} = 0$ and $J^k_{0*} = \emptyset \ \forall k \in K$.

4.1.8. *Collateral Equilibrium*

A *Collateral Equilibrium* in this economy is a set of consumption goods prices, financial asset prices and contract prices, production and consumption decisions, and financial decisions on assets and contract holdings $((q, p, \pi), (z^h, c^h, y^h, \varphi^h)_{h \in H}) \in (R^L_+)_{s \in S} \times (R^K_+ \times R^{J_s}_+)_{s \in S \setminus S_T} \times (R^{S(L+K)} \times R^{SL}_+ \times R^{SK}_+ \times (R^{J_s})_{s \in S \setminus S_T})^H$ such that

1. $\sum_{h \in H}(c^h_s - e^h_s - F^h_s(c_{s*}) - z^h_{sc}) = \sum_{h \in H} \sum_{k \in K} y^h_{s*k} d^k_s, \forall s.$
2. $\sum_{h \in H}(y^h_s - y^h_{s*} - z^h_{sy}) = 0, \forall s \in S \setminus S_T.$
3. $\sum_{h \in H} \varphi^h_j = 0, \forall j \in J_s, \forall s.$
4. $(z^h, c^h, y^h, \varphi^h) \in B^h(q, p, \pi), \forall h.$
5. $(z, c, y, \varphi) \in B^h(q, p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h.$

Markets for consumption, assets, and promises clear in equilibrium and agents optimize their utility in their budget set.

4.2. *General No-Default Theorem*

It turns out that we can still assume no default in equilibrium without loss of generality in this much more general context, as the following theorem shows.

BINOMIAL NO-DEFAULT THEOREM: *Suppose that S is a binomial tree, that is, $S(s) = \{sU, sD\}$ for each $s \in S \setminus S_T$. Suppose that all assets are financial assets and that every contract is a one-period contract. Let $((q, p, \pi), (z^h, c^h, y^h, \varphi^h)_{h \in H})$ be an equilibrium. Suppose that for any state $s \in S \setminus S_T$, any asset $k \in K$, and any contract $j \in J_s^k$, the max min promise $(\bar{\lambda} j_{sU}, \bar{\lambda} j_{sD})$ is available to be traded, where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(q_{sU} \cdot j_{sU}, q_{sD} \cdot j_{sD}) \leq (p_{sUk} + q_{sU} \cdot d_{sU}, p_{sDk} + q_{sD} \cdot d_{sD})\}$. Then we can construct another equilibrium $((q, p, \pi), (z^h, c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same asset and contract prices and the same production and consumption choices, in which only max min contracts are traded.*

PROOF: The proof of the Binomial No-Default Theorem can be applied in this more general context state by state, asset by asset, and ray by ray. Take any $s \in S \setminus S_T$ and any asset $k \in K$. Partition J_s^k into $J_s^k(r_1) \cup \cdots \cup J_s^k(r_n)$ where the r_i are distinct rays $(\mu_i, \nu_i) \in \mathbb{R}_+^2$ of norm 1 such that $j \in J_s^k(r_i)$ if and only if $(q_{sU} \cdot j_{sU}, q_{sD} \cdot j_{sD}) = \lambda(\mu_i, \nu_i)$ for some $\lambda > 0$. For each agent $h \in H$, consider the portfolio $(y^h(s, k, i), \varphi^h(s, k, i))$ defined by

$$\varphi_j^h(s, k, i) = \varphi_{sj}^h \quad \text{if} \quad j \in J_s^k(r_i) \quad \text{and} \quad 0 \quad \text{otherwise,}$$

$$y^h(s, k, i) = \sum_{j \in J_s^k(r_i)} \max(0, \varphi_{sj}^h).$$

Denote the portfolio payoffs in each state by

$$w_U^h(s, k, i) = y^h(s, k, i)[p_{sUk} + q_{sU} d_{sU}^k]$$
$$- \sum_{j \in J_s^k(r_i)} \varphi_j^h(s, k, i) \min(q_{sU} \cdot j_{sU}, p_{sUk} + q_{sU} d_{sU}^k),$$

$$w_D^h(s, k, i) = y^h(s, k, i)[p_{sDk} + q_{sD} d_{sD}^k]$$
$$- \sum_{j \in J_s^k(r_i)} \varphi_j^h(s, k, i) \min(q_{sD} \cdot j_{sD}, p_{sDk} + q_{sD} d_{sD}^k).$$

If

$$\frac{\mu_i}{\nu_i} < \frac{p_{sUk} + q_{sU} d_{sU}^k}{p_{sDk} + q_{sD} d_{sD}^k},$$

then the combination of the Arrow U security (which can be obtained by buying the asset k while borrowing on the max min contract of type (s, k, i)) and the max min contract of type (s, k, i) positively spans $(w_U^h(s, k, i), w_D^h(s, k, i))$. Thus we can apply the proof of the Binomial No-Default Theorem to replace

all the above trades of contracts in $J_s^k(r_i)$ with a single trade of the max min contract of type (s, k, i). If

$$\frac{\mu_i}{\nu_i} > \frac{q_{sUk} + p_{sU}d_{sU}^k}{q_{sDk} + p_{sD}d_{sD}^k},$$

then exactly the same logic of the Binomial No-Default Theorem applies, but with the Arrow D security instead of the Arrow U security. If there is equality in the above comparison, then the contract and the asset are perfect substitutes, so there is no need to trade the contracts in the family at all. *Q.E.D.*

4.3. *Discussion*

The generalized Binomial Default Theorem applies in a far broader context than the theorem stated in Section 3. The reason is that the proof of Section 3 did not use the fact that the promises were noncontingent, but only that they all were on the same ray. Furthermore, by assumption an asset can only back one loan, so the total collateral backing promises along some ray can be considered separately from the rest of the assets. Since in addition, all promises are one period long, the simple proof of Section 3 can be applied state by state to each asset and each homogeneous family of promises (i.e., promises along a single ray), using the asset as collateral. As in the proof in Section 3, borrowers can use less of this collateral to achieve the same final payoffs at the same cost by using only the max min contract. Thus our two refinements in Section 3.3 also extend to this general setting. Any positive fee for collateral use guarantees that, in every equilibrium, only max min contracts are traded. It may now be the case that sometimes the payoff cone is given by the positive span of the max min of the family and the Arrow D security, instead of the Arrow U security. However, the logic of the argument stays completely unaltered.

On the other hand, the State Pricing Corollary of Section 3.1 does not extend to this more general context. For each ray, say r_i, we obtain (by the same logic as before) state prices a_i and b_i. However, they need not be the same as the state prices obtained when the argument is applied to a different ray, say r_j. The reason for this is that the payoff cones associated to each ray may not completely coincide. Hence, we only have a "local" State Pricing result.

The generalization shows that the No-Default Theorem holds in an important setting. But it also allows us to reach some more conclusions. First, let us consider what we learn from multiple assets k. Suppose for now that we restrict attention to noncontingent contracts, that is, contracts $j \in J_s^k$ for which, in equilibrium, $q_{sU} \cdot j_{sU} = q_{sD} \cdot j_{sD}$.[17] The Binomial Leverage Theorem presented

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[17]These may be created by assuming there is some numeraire bundle of goods v_s such that commodity prices always satisfy $q_s \cdot v_s = 1$ and then supposing that promises are denoted in units of the numeraire. In the actual world, many contract promises are denoted by noncontingent money payments.

in Section 3.4 extends to many assets. Letting LTV_s^k denote the leverage of every riskless loan collateralized by k in state s, we must have

$$LTV_s^k = \frac{\min\{p_{sD} + q_{sD} \cdot d_{sD}^k, \, p_{sU} + q_{sU} \cdot d_{sU}^k\}/p_{sk}}{1 + r_s},$$

where $(1 + r_s) = q_{sU} \cdot j_{sU}/\pi_j$ for any noncontingent contract $j \in J_s$ whose deliveries are fully covered by the collateral. This formula explains which assets are easier to leverage. The asset whose future value has the least bad downside can be leveraged the most. The formula allows us to rank leverage of different assets at the same state s, or even across states and across economies. When contract promises are contingent, the Binomial No-Default Theorem tells us exactly how big the promises of each type will be made per unit of collateral: as big as can be guaranteed not to default. But the leverage formula for the LTV associated to noncontingent contracts cannot so easily be extended to contingent contract promises, because it depends on what the state prices are.

Many papers have assumed a link between volatility and leverage (see, e.g., Thurner, Farmer, and Geanakoplos (2012) and Adrian and Boyarchenko (2012)). The Binomial Leverage Theorem indeed shows that as the world becomes more dangerous, leverage goes down. But it also makes clear that volatility is not, in general, an appropriate measure of danger, except when asset payoffs are symmetric (state prices of 1/2, 1/2) or when all asset payoffs are perfectly correlated. Consider two assets k and k' that are both traded in some state s. Suppose that their one-period asset returns are perfectly correlated, so they both have a lower return in sD than in sU. Suppose that the down risk for k' is worse than the down risk for k, so that k can be leveraged more. Clearly, the return to k' at U must be higher than the return to k at U; otherwise, nobody would have held k' (for then its returns would be strictly dominated and it would be less good collateral). Hence, no matter what probabilities $a > 0$ and $b > 0$ we attach to the two states, a dollar's worth of k' has more volatility than a dollar's worth of k. But if the assets are not perfectly correlated, this argument fails. Suppose now that both assets and their bonds can be priced by the same state prices $a = 2/3$, $b = 1/3$. Suppose their payoffs are $d_U^k = 4$, $d_D^k = 1$, $d_U^{k'} = 2$, $d_D^{k'} = 5$. Then both assets have the same price $p_{sk} = p_{sk'} = 3$, and at probabilities $(a, b) = (2/3, 1/3)$, both have the same volatility. But k' has less downside risk and so can be leveraged more. In the binomial world, down risk, not volatility, determines leverage.

The Binomial No-Default Theorem compares contracts along the same ray, determining which will be traded. In this paper, we do not pursue the question of comparing contracts along different rays. However, notice that our proof shows that the collateral minimizing equilibrium always uses the rays which, together with the asset payoffs, generate the biggest positive span. At most two contracts will be traded with the same collateral. In particular, if Arrow securities can be issued, they will be. This result about Arrow securities already appears in the unpublished work of Geanakoplos and Zame.

The general setting described in Section 4 allows for idiosyncratic intertemporal production of commodities. This is an important extension. Allowing for production in collateral general equilibrium makes it possible to study the effect of an upward sloping Credit Surface on investment. Consider an agent h who can produce two apples in sU and also in sD with an input of just one apple in s. She may be holding collateral in state s that she likes (say because she is optimistic and it pays more in sU than in sD) and that she leverages at zero rate of interest. To the outside observer, it may appear that the rate of interest in the economy is zero, and that nobody is undertaking any investment because nobody can find an investment with a positive rate of return. But that would be wrong. Agent h indeed has a much more productive investment. But she cannot borrow the money to undertake it. If she tries to borrow more money on her asset, the interest rate will go up so fast that it is not worth it. Furthermore, she does not want to sell the asset because she is getting to buy it for such a low price. We pursue these kind of questions in Fostel and Geanakoplos (2016).

The extension to trees of arbitrary periods allows us to see why Brownian motion economies are not the appropriate models to study endogenous leverage. By taking shorter and shorter time periods, binomial economies converge to Brownian motion economies, provided certain conditions are met. We do not have room to spell out all the details, but one condition (familiar from the Black–Scholes literature) is that as the period grows shorter, the one-period volatility and down risk of the asset price go to zero. From the Binomial Leverage Theorem, we conclude that the LTV on the shorter and shorter one-period loans converges to 100%. Thus in the limit, investors with a tiny bit of money can buy all the assets they want without running into a collateral constraint. In the limit, leverage is entirely demand-determined. However, the limiting equilibrium could still be far from Arrow–Debreu. Agents cannot sell contingent promises, and they cannot borrow to consume (since even with 100% LTV loans, the collateral they hold is at least as costly as the money they obtain by borrowing).[18]

4.4. *Pyramiding and Binomial Assets*

One might wonder whether the same theorem would hold if we allowed the contracts to serve in turn as collateral for further contracts. Houses are used as collateral for mortgages, and mortgages (after they are bundled together as securities) are used as collateral for further loans like Repos. For simplicity, let us restrict attention to noncontingent promises. Suppose an agent i issues a contract $j > j^* = d_D$ backed by an asset paying $d_U > j > d_D$, and then another agent i' uses the j contract as collateral to issue his own promise of j' to some agent i''. A moment's reflection reveals that the contracts j, j' are in the positive span of the asset and j^*. Hence exactly the same proof can be applied to derive the State Pricing Corollary and then the Binomial No-Default Theorem in this more general setting.

[18]See, for example, Brunnermeier and Sannikov (2013).

The No-Default Theorem allows for two further extensions, both showing that it really depends on binomial assets rather than binomial trees. First, it can be extended to more than two successor states, provided that, for each financial asset, the states can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant. Second, it can also be extended to contracts with longer maturities. Suppose all the contracts written on some financial asset come due in the same period and that the states in that period can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant. Suppose also that the financial asset used as collateral cannot be traded or used for production purposes before maturity. Then the proof of the Binomial No-Default Theorem shows that, without loss of generality, we can assume no default in equilibrium.

5. BINOMIAL CAPM EXAMPLES

In this section, we present two examples to illustrate our theoretical results. We assume one perishable consumption good and one asset which pays dividends $d_U > d_D$ of the consumption good. Consider two types of mean–variance investors, $h = A, B$, characterized by utilities $U^h = u^h(c_0) + \sum_{s \in S_T} \gamma_s u^h(c_s)$, where $u^h(c_s) = c_s - \frac{1}{2} \alpha^h c_s^2$, $s \in \{0, U, D\}$. Agents do not discount the future. Agents have an initial endowment of the asset, y_{0*}^h, $h = A, B$. They also have endowment of the consumption good in each state, e_s^h, $\forall s, h = A, B$. It is assumed that all contract promises are of the form (j, j), $j \in J$, each backed by one unit of the asset as collateral. Agents will never deliver on a promise beyond the value of the collateral since we assume nonrecourse loans.[19]

5.1. *Binomial CAPM With Multiple Equilibria*

Suppose agents start with endowment of the asset, $y_{0*}^A = 1$, $y_{0*}^B = 3$. Suppose endowments of the consumption good are given by $e^A = (e_0^A, (e_U^A, e_D^A)) = (1, (1, 5))$ and $e^B = (e_0^B, (e_U^B, e_D^B)) = (3, (5, 5))$. Utility parameters are given by $\gamma_U = \gamma_D = 0.5$ and $\alpha^A = 0.1$ and $\alpha^B = 0.1$. Finally, asset payoffs are $d_U = 1$ and $d_D = 0.2$. Type-A agents have a tremendous desire to buy Arrow U securities and present consumption, and to sell Arrow D securities. But they are limited by the restriction to noncontingent contract promises (j, j).

According to the Binomial No-Default Theorem, in searching for equilibrium we never need to look beyond the max min promise $j^* = 0.2$, for which

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[19]This example would satisfy all the assumptions of the classical CAPM (extended to untraded endowments), provided that we assumed agents always kept their promises, without the need of posting collateral.

TABLE I

COLLATERAL EQUILIBRIUM WITH NO DEFAULT: PRICES AND LEVERAGE

Variable	Notation	Value
Asset price	p	0.3778
State price	a	0.3125
State price	b	0.3264
Max min contract price	π_{j*}	0.1278
Leverage	LTV^Y	0.3382

there is no default. Tables I and II present this max min collateral equilibrium. Type-A agents buy most of the assets in the economy, $y^A = 3.7763$, and use their holdings as collateral to sell the max min contract, promising $(0.2)(3.7763)$ in both states U and D. Type-B investors sell most of their asset endowment and lend to type-A investors.[20]

As indicated by the Binomial State Pricing Corollary, all the contracts $j \neq j^*$, as well as $j = j^*$, can be priced by state prices $a = 0.3125$ and $b = 0.3264$. By the No-Default Theorem, we do not need to investigate trading in any of the contracts $j \neq j^*$. Indeed, it is easy to check that this is a genuine equilibrium, and that no agent would wish to trade any of these contracts $j \neq j^*$ at the

TABLE II

COLLATERAL EQUILIBRIUM WITH NO DEFAULT: ALLOCATIONS

	Asset and Collateral	
	Asset y	Contracts φ_{j*}
Type-A	3.7763	3.7763
Type-B	0.2237	−3.7763
	Consumption	
	$s = 0$	$s = U$	$s = D$
Type-A	0.4337	4.0211	5
Type-B	3.5663	5.9789	5.80

[20] To find the equilibrium, we guess the regime first and we solve for three variables, p, π_{j*}, and ϕ_{j*}, a system of three equations. The first equation is the first-order condition for lending corresponding to the B-investor: $\pi = \frac{q_U(1-\alpha^B c_U^B)d_D + q_D(1-\alpha^B c_D^B)d_D}{1-\alpha^B c_0^B}$. The second equation is the first-order condition of the A-investor for purchasing the asset via the max min contract, $p - \pi = \frac{q_U(1-\alpha^A c_U^A)(d_U-d_D) + q_D(1-\alpha^A c_D^A)(d_D-d_D)}{1-\alpha^A c_0^A}$. The third equation is the first-order condition for the B-investor for holding the asset, $p = \frac{\gamma_U(1-\alpha^B c_U^B)d_U + \gamma_D(1-\alpha^B c_D^B)d_U}{1-\alpha^B c_0^B}$. Finally, we check that the regime is genuine, confirming that the A-investor really wants to leverage to the max; for this to be the case, $\pi > \frac{\gamma_U(1-\alpha^A c_U^A)d_D + \gamma_D(1-\alpha^A c_D^A)d_D}{1-\alpha^A c_0^A}$.

TABLE III

COLLATERAL EQUILIBRIUM WITH DEFAULT: PRICES AND LEVERAGE

Variable	Notation	Value
Asset price	p	0.3778
Promise	j	0.2447
Contract j price	π_j	0.1418
Leverage	LTV^Y	0.3753

prices given by a, b. Every agent who leverages chooses to sell the same max min contract; hence asset leverage and contract leverage are the same and described in the table. We can easily check that the LTV^Y satisfies the formula given in the Binomial Leverage Theorem, so that

$$LTV^Y = \frac{d_D/p}{1 + r_{j^*}} = \frac{0.2/0.3778}{1.56} = 0.3382.$$

This equilibrium is essentially unique, but not strictly unique. In fact, there is another equilibrium *with* default as shown in Tables III and IV, in which the type-A agents borrow by selling the contract $j = 0.2447 > j^* = 0.2$. In the default equilibrium, leverage is higher and the asset holdings of type-A agents are higher (so diluted leverage is much higher). They borrow more money. However, as guaranteed by the Binomial No-Default Theorem, in both equilibria, consumption and asset and contract prices are the same: *actual* default is irrelevant. Notice that in the no-default equilibrium, 3.7763 units of the asset are used as collateral, while in the default equilibrium, 4 units of the asset are used as collateral. The no-default equilibrium uses less collateral.

Between these two equilibria, the Modigliani–Miller Theorem holds; there is an indeterminacy of debt issuance in equilibrium. However, leverage cannot

TABLE IV

COLLATERAL EQUILIBRIUM WITH DEFAULT: ALLOCATIONS

	Asset and Collateral	
	Asset y	Contracts φ_j
Type-A	4	4
Type-B	0	−4
	Consumption		
	$s = 0$	$s = U$	$s = D$
Type-A	0.4337	4.0211	5
Type-B	3.5663	5.9789	5.80

TABLE V

ARROW–DEBREU AND CAPM EQUILIBRIUM

	$s = 0$		$s = U$	$s = D$
Asset price	p		0.3700	
State price	p_U		0.3125	
State price	p_D		0.2875	
			Consumption	
Type-A	0.5338		4.0836	4.5569
Type-B	3.4662		5.9164	6.2431
			CAPM Portfolios	
			Market	Bond
Type-A			0.5916	1.8328
Type-B			0.4084	−1.8328

be reduced below the max min contract level. If type-A agents were forced to issue still less debt, they would be strictly worse off. Thus in this example, the No-Default Theorem holds while the Modigliani–Miller Theorem fails beyond a limited range.

Finally, both collateral equilibria are different from the Arrow–Debreu Equilibrium and the classical CAPM equilibrium, as shown in Table V.

State prices in collateral equilibrium are different from the state prices in Arrow–Debreu equilibrium. The asset price in complete markets is slightly lower than in collateral equilibrium. In the complete markets equilibrium, investors hold shares in the market portfolio $(10, 10.8)$ (aggregate endowment) and in the riskless asset $(1, 1)$.

5.2. Binomial CAPM With a Unique Equilibrium

Consider the same CAPM model as in Section 5.1 except that now $y_{0*}^h = 1$, $h = A, B$.

Tables VI and VII present the max min collateral equilibrium. In the collateral equilibrium, type-A agents buy all the asset in the economy and use all of their holdings as collateral, leveraging via the max min contract. On the other hand, type-B investors sell all their asset endowment and lend. As before, LTV^Y is characterized by down risk.

Unlike the previous example, the no-default equilibrium in this example is unique without any need of refinements. We cannot find another equilibrium involving default with borrowers issuing bigger promises, since there is not enough collateral in the economy. In this case, as before, the collateral equilibrium does not coincide with the complete markets equilibrium shown in Table VIII.

TABLE VI

COLLATERAL EQUILIBRIUM WITH NO DEFAULT: PRICES AND LEVERAGE

Variable	Notation	Value
Asset price	p	0.4572
State price	a	0.4027
State price	b	0.2725
Max min contract price	π_{j*}	0.1350
Leverage	LTV^Y	0.2952

TABLE VII

COLLATERAL EQUILIBRIUM WITH NO DEFAULT: ALLOCATIONS

	Asset and Collateral	
	Asset y	Contracts φ_{j*}
Type-A	2	2
Type-B	0	−2
	Consumption	
	$s = 0$	$s = U$	$s = D$
Type-A	0.8122	2.6	5
Type-B	3.1872	5.4	5.4

TABLE VIII

ARROW–DEBREU AND CAPM EQUILIBRIUM

Asset price	p	0.4350	
State price	p_U	0.3750	
State price	p_D	0.3	
	Consumption		
	$s = 0$	$s = U$	$s = D$
Type-A	0.8024	3.1018	4.4814
Type-B	3.1976	4.8982	5.9186
	CAPM Portfolios		
	Market		Bond
Type-A	0.5749		1.4970
Type-B	0.4251		−1.4970

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